How far can the Rödl nibble go?

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Joint work with Stephen Gould

Combinatorics at the Confluence
Carnegie Mellon University, University of Pittsburgh
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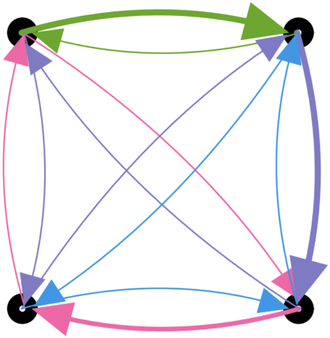
Part I

Applications

Let G be a directed graph.

proper arc-coloring: at each vertex, all outgoing arcs receive distinct colors, and all incoming arcs receive distinct colors

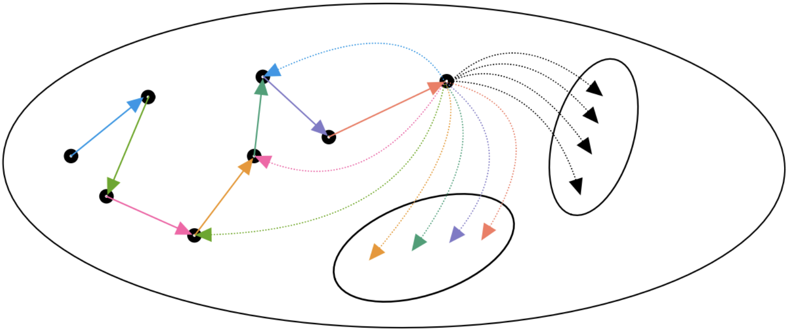
rainbow: edges have distinct colors



A rainbow path of length three in a proper arc-coloring of $K_4^{\leftrightarrow}$.

Question

Given a proper n-arc-coloring of $K_n^{\leftrightarrow}$, what is the expected length of a rainbow self-avoiding walk constructed by random greedy (as $n \to \infty$)?



Question

Given a proper n-arc-coloring of $K_n^{\leftrightarrow}$, what is the expected length of a rainbow self-avoiding walk constructed by random greedy (as $n \to \infty$)?

Heuristics suggest: $n - \sqrt{n}$

Let $p = (1 - t/n)$. At time t, pn vertices and colors remain unused, and they should "look like" a set chosen independently with probability p.

In the independent setting,

$$\mathbb{E}\left[\# \text{ valid choices}\right] = p^2 n,$$

and

$$p^2 n < 1 \qquad \Rightarrow \qquad p < 1/\sqrt{n}.$$

Conjecture (Gyárfás & Sárközy, 2014;, Gould & Kelly, 2023)

Every properly arc-colored $K_n^{\leftrightarrow}$ contains a rainbow path or cycle of length at least $n - 1$.

- If true, implies both **Andersen's conjecture (2005)** and the even-n case of the **Ryser–Brualdi–Stein conjecture (1967)**.

Benzing, Pokrovskiy, and Sudakov (2020): $n - O(n^{4/5})$

Balogh and Molla (2019): $n - O(\log n\sqrt{n})$ for undirected graphs

Gould and Kelly (2023): true for *almost all* proper n-arc-colorings

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Theorem (Gould and Kelly, 2025+)

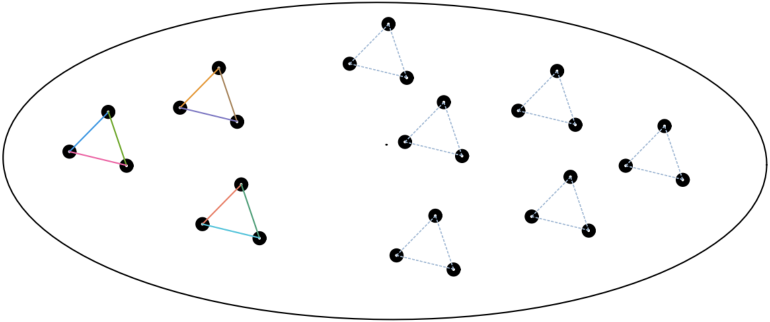
Every properly n-arc-colored $K_n^{\leftrightarrow}$ contains a rainbow cycle of length at least $n - n^{1/2+o(1)}$.

- probabilistic proof – matches heuristic

Conjecture (Müyesser, 2023)

Every properly n-edge-colored K_n contains a rainbow triangle tiling covering all but $O(1)$ vertices.

- Motivated by **Friedlander-Gordon-Tannenbaum conjecture (1981)** from Group Theory.



Conjecture (Müyesser, 2023)

Every properly n-edge-colored K_n contains a rainbow triangle tiling covering all but $O(1)$ vertices.

Question

What is the expected number of vertices left uncovered by a rainbow triangle tiling constructed by random greedy (as $n \to \infty$)?

Heuristics suggest: $n^{3/5}$

If vtcs and colors are chosen independently with prob. p, then $\forall$vertex v,

$$\mathbb{E}\left[\# \text{ triangles containing } v\right] = p^5 \binom{n}{2},$$

and

$$p^5 n^2 < 1 \qquad \Rightarrow \qquad p < 1/n^{2/5}.$$

Conjecture (Müyesser, 2023)

Every properly n-edge-colored K_n contains a rainbow triangle tiling covering all but $O(1)$ vertices.

Theorem (Gould and Kelly, 2025+)

Every properly n-edge-colored K_n has a rainbow triangle tiling covering all but at most $n^{3/5+o(1)}$ vertices.

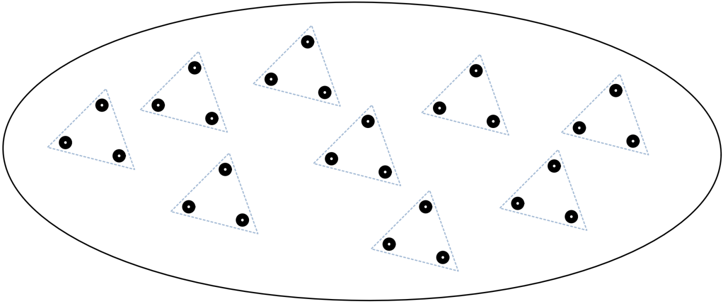
- The best that can be obtained with prior results is $n^{3/4+o(1)}$.

Part II

Hypergraph matchings

matching: set of pairwise disjoint edges

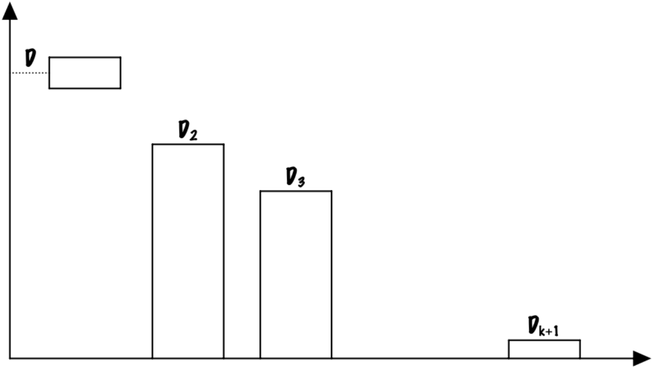
k-uniform: every edge has size k



A 3-uniform matching

In what follows, H is an n-vertex $(k+1)$-uniform hypergraph which is

- $(1 \pm \varepsilon)D$-regular, and
- every set of i vertices is contained in at most D_i edges of H.



Pippenger's Theorem (Füredi, 1988)

If $\varepsilon \ll \gamma$ and $D_2 \leq \varepsilon D$, then H has a matching covering all but $\leq \gamma n$ vtcs.

Heuristic: $D_2 \ll D \Rightarrow$ when pn vertices remain in nibble/random greedy, vertices "look like" they are chosen independently with probability p.

In the independent setting,

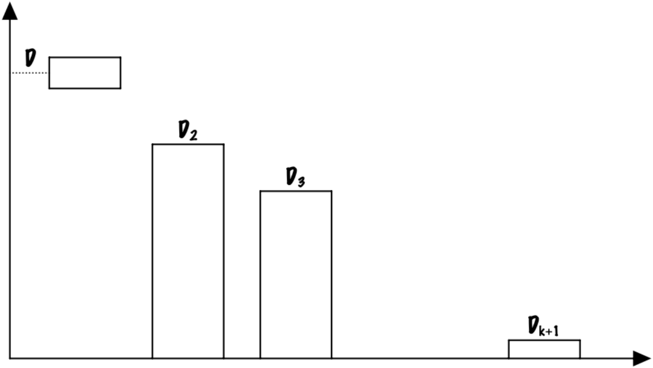
$$\mathbb{E}\left[\text{degree of } v\right] = p^k D,$$

and

$$p^k D \gg D_2 \qquad \Rightarrow \qquad p \gg (D_2/D)^{1/k}$$

Theorem (Kostochka and Rödl, 1997)

If $\varepsilon \leq O(\sqrt{\log D / D})$ and $D_2 \leq D^{1-o(1)}$, then H has a matching covering all but $\leq (D_2/D)^{1/k+o(1)} n$ vtcs.



The codegree sequence before nibble

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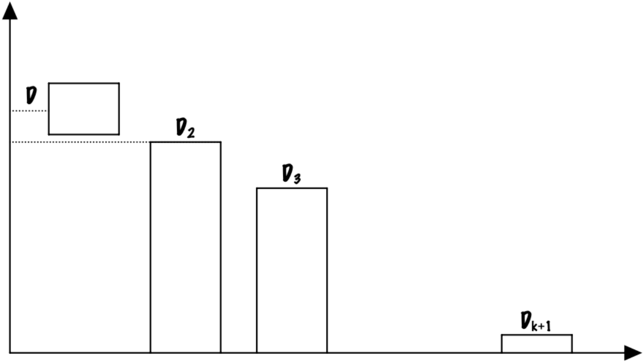


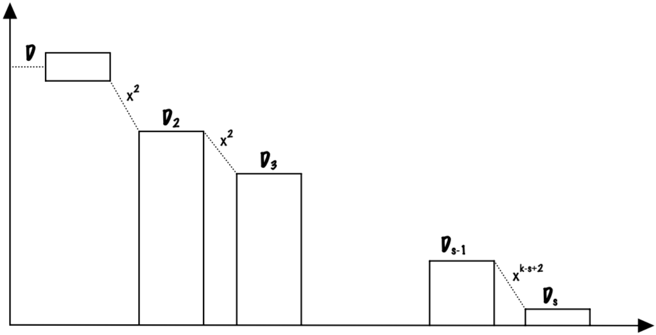
The codegree sequence after nibble

Theorem (Vu, 2000)

If there exists $x > 0$ and $s \in [k+1]$ such that

- (**well-spaced codegrees**) $x^2 \leq D_j/D_{j+1}$ for all $j \in [s-1]$;
- (**large penultimate codegree gap**) $x^{k-s+2} \leq D_{s-1}/D_s$;
- (**nearly regular**) $\varepsilon \leq 1/x$,

then H has a matching covering all but $\leq n(\log^{O(1)} D)/x$ vtcs.



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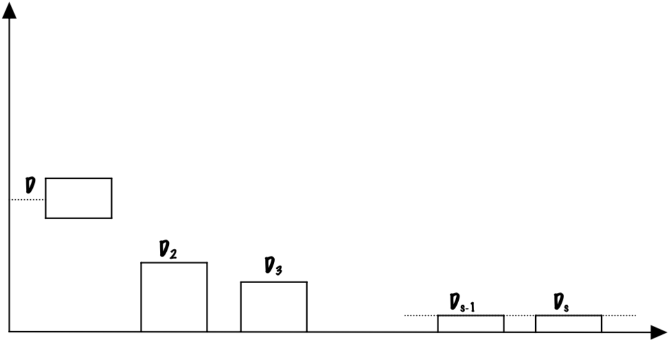

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Our idea: Can we iterate? Reapply Vu with $s \leftarrow s-1$?

Two main issues:

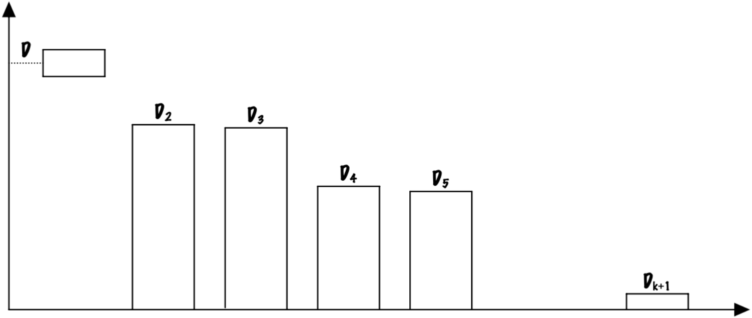
- The remaining hypergraph is not sufficiently regular.
- Vu's result cannot handle "clustered codegrees".

Theorem (Gould and Kelly, 2025+)

If there exists $B > 0$ such that

- **(large degree/codegree ratios)** $B \leq (D/D_j)^{\frac{1}{j-1}}\ \forall j \geq 4$;
- **(large 2-degree gap)** $B \leq \sqrt{D/D_2}$;
- **(nearly regular)** $\varepsilon \leq 1/B$,

then H has a matching covering all but $\leq n(\log^{O(1)} D)/B^{1-o(1)}$ vtcs.



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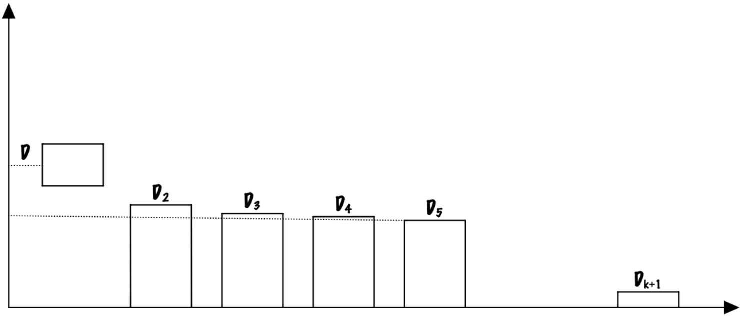

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Heuristic: $D_{j+1} \ll D_j \Rightarrow$ when pn vertices remain in nibble/random greedy:

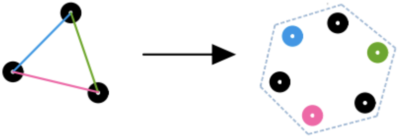
$$\mathbb{E}\left[\text{degree of a } j\text{-set}\right] \leq p^{k+1-j} D_j,$$

and

$$p^{k+1-j} D_j \ll p^k D \qquad \Rightarrow \qquad p \gg (D_j/D)^{\frac{1}{j-1}}$$

Theorem (Gould and Kelly, 2025+)

Every properly n-edge-colored K_n has a rainbow triangle tiling covering all but at most $n^{3/5+o(1)}$ vertices.



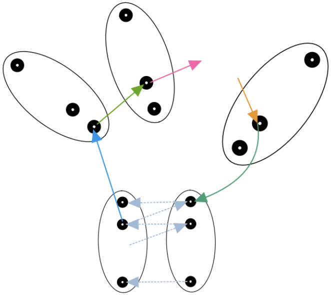
$$\text{Rainbow triangle tilings} \qquad \longleftrightarrow \qquad \text{matchings in } H,$$

where H is 6-uniform, $\binom{n}{2}$-regular, with codegree sequence

$$\left(\binom{n}{2}, n-2, n-2, 1, 1\right)$$

Theorem (Gould and Kelly, 2025+)

Every properly n-arc-colored $K_n^{\leftrightarrow}$ contains a rainbow cycle of length at least $n - n^{1/2+o(1)}$.



Long rainbow cycles $\longleftarrow$ "A-perfect" matchings in H,

where H is $2\ell + 1$-uniform, $\approx c_\ell n^\ell$-regular, with codegree sequence

$$\left(c_\ell n^\ell, n^{\ell-1}, n^{\ell-1}, \ldots, n, n, 1, 1, 1\right).$$

Other applications of our result: Improvements to

- **Molloy–Reed (2000)** bound on chromatic index of uniform hypergraphs that are far from linear;
- **Bohman–Newman (2022)** asymptotics for maximum diameter of a d-dimensional strongly connected simplicial complex:
  - **Glock–Parczyk–Rathke–Szabó (2026+):** tight bound;
- **Kang–Kühn–Methuku–Osthus (2023)** bound on the size of a maximum matching and the chromatic index of an (n, q, r)-Steiner system for $r > 2$.

Tiers of generality for nearly perfect hypergraph matchings:

- pseudorandom nearly perfect hypergraph matchings:
  - **Ehard–Glock–Joos (2020)**; **Alon–Yuster (2005)**;
- tripartite/reserve perfect matchings:
  - **Delcourt–Postle (2024+)**; **Joos–Mubayi–Smith (2024+)** ;
- bipartite "A-perfect matchings":
  - **Delcourt–Postle (2022+)**;
- asymptotically good list edge-coloring:
  - **Kahn (1996)**;
- asymptotically good proper edge-coloring:
  - **Pippenger–Spencer (1989)**;
- nearly perfect matchings:
  - **Frankl–Rödl (1985)**; **Pippenger (1988)**.

Thanks for listening!