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Geodesic complexity of the octahedron

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Geodesic complexity (it's actually quite simplex)

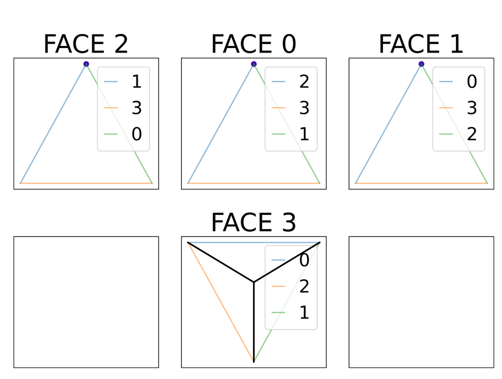
Topological complexity [2] describes the difficulty of creating a continuous motion plan over a space X. X has topological complexity k (denoted $TC(X) = k$) if k is the minimal number such that there is an open cover $U_1 \cup U_2 \cup \cdots \cup U_k = X \times X$ where each U_i has a local ***motion planning rule***: a continuous map $\sigma_i \colon U_i \to PX$ (with PX the space of paths in X) such that $\sigma_i(x, y)$ is a path from x to y.

Geodesic complexity [5] is analogous to topological complexity, with the caveat that all paths are ***geodesics*** (shortest paths with constant speed). $GC(X) = k$ if k is the minimal number such that there is a *locally compact partition* $E_0 \sqcup E_1 \sqcup \cdots \sqcup E_k = X \times X$ where each E_i has a local ***Geodesic Motion Planning Rule (GMPR)*** $\sigma_i \colon E_i \to GX$. Here, $GX \subseteq PX$ is the space of geodesics in X. We call $\sigma_0, \ldots, \sigma_k$ a ***geodesic motion planner***.

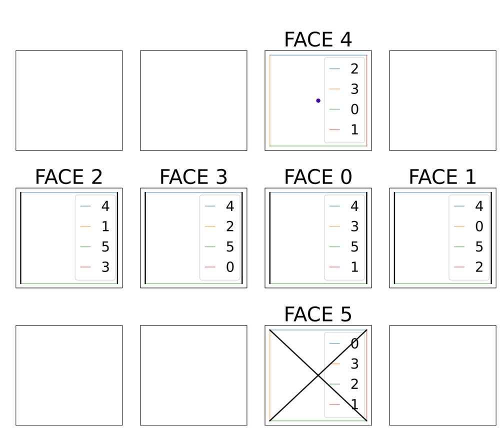
Cut locus and its computation

The ***cut locus*** of a point $p \in X$ is the set of all points in X with multiple distinct geodesics to p. The structure of the cut loci in X is vital to the computation of $GC(X)$, as it is exactly the set of points where a GMPR must make a decision.

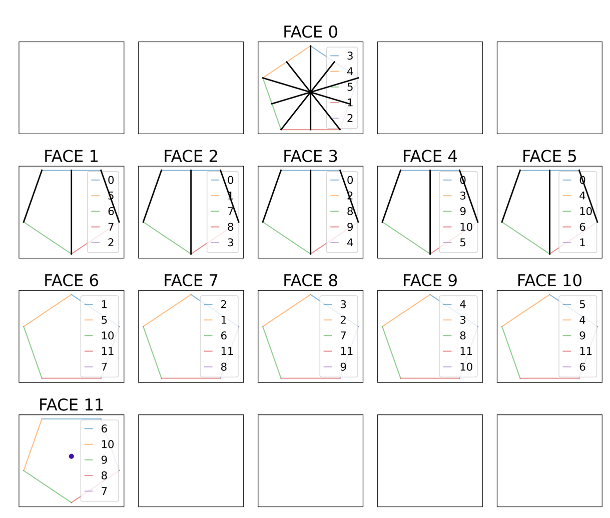
For X the surface of a polyhedron (with the distance between points as the length of the shortest path between them), there exists a polynomial-time algorithm to compute the cut locus of a given point [4]. Using this algorithm, we show that the geodesic complexity of the octahedron is four.



(a) Tetrahedron



(b) Cube



(c) Dodecahedron

Figure 1. Cut locus of point • on a few polyhedra.

Theorem (simplex embedding theorem)

For integers $0 \leq k_0 < k$, if for each $i \in \{k_0, k_0+1, \ldots, k\}$ there exists a non-empty collection $\mathcal{F}_i$ of embeddings into $X \times X$ such that:

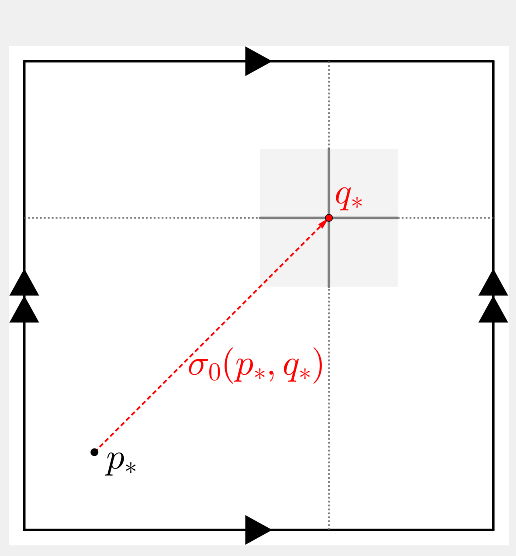
(a) Each $\mathcal{F}_i$ consists of embeddings of i-dimensional simplices $\Delta_i \hookrightarrow X \times X$.

(b) For $F \in \mathcal{F}_i$, any geodesic in $\pi_{GX}^{-1}(\text{relint}^*(F))$ extends to a GMPR on all $\text{img}(F)$.
(Here, $\text{relint}^*(F)$ is the ***relative interior*** of $\text{img}(F)$, and $\pi_{GX} \colon GX \to X \times X$ maps a geodesic to its start and end point.)

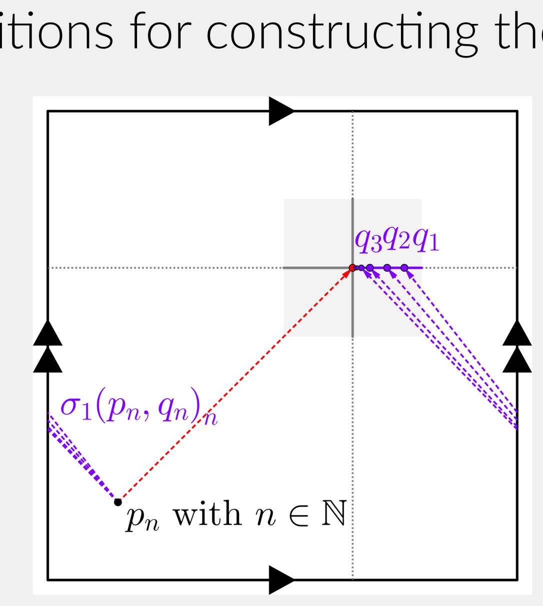
(c) With $k_0 \leq i < k$, for any $F \in \mathcal{F}_i$ and GMPR Γ on $\text{img}(F)$, there is an $F' \in \mathcal{F}_{i+1}$ such that:
   - *(i) The map F is the restriction of F' to a face of Δ_{i+1}.*
   - *(ii) Each geodesic in $\Gamma(\text{img}(F))$ is separated from any geodesics in $\pi_{GX}^{-1}(\text{relint}^*(F'))$ by an open set.*
   - *(iii) There are a finite number of GMPRs on $\text{img}(F')$.*

Then $GC(X) \geq k - k_0$ (i.e. any geodesic motion planner requires $\geq k - k_0 + 1$ sets).

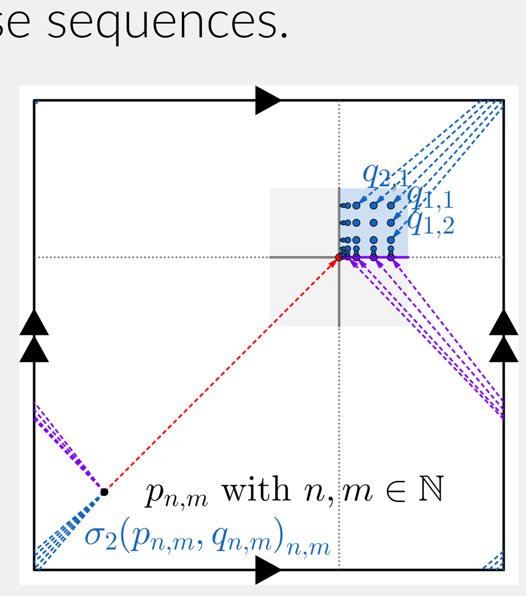
This method is inspired by arguments of Davis [1] and Recio-Mitter [5], whose proofs use sequences, sequences of sequences, etc. of elements in $X \times X$. Each iteration is constructed so it must be in a unique GMPR. Our theorem describes sufficient conditions for constructing these sequences.



(a) Base case $(p_*, q_*) \in E_0$ and possible GMPR σ_0



(b) Choice of $F_1 \in \mathcal{F}_1$, sequence $(p_n, q_n)_n \subseteq E_1$, and possible σ_1



(c) $F_2 \in \mathcal{F}_2$, $(p_{n,m}, q_{n,m})_{n,m} \subseteq E_2$, and possible σ_2

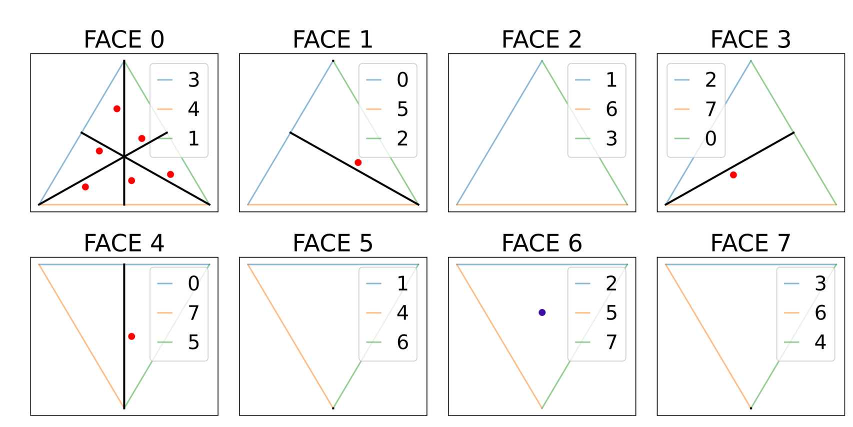
Figure 2. Mechanism of theorem when used on the 2-torus. Each embedding $F \in \mathcal{F}_1$ (resp. $F' \in \mathcal{F}_2$) embeds a simplex into $\{p_*\} \times X$ by selecting the second factor to map to one of the shaded lines (resp. squares). F_1 (resp. F_2) is chosen to avoid $\sigma_0(p_*, q_*)$ (resp. $\sigma_1(p_n, q_n)_n$).

Upper bound on GC(octahedron)

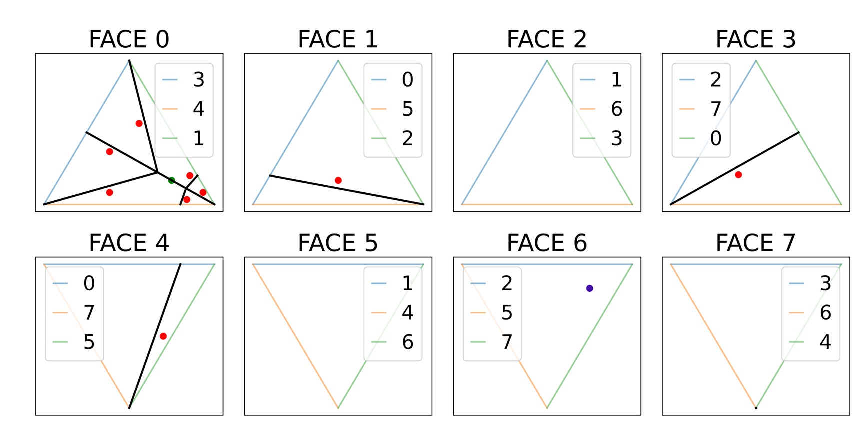
We show $GC(\text{octahedron}) \leq 4$ by constructing an explicit geodesic motion planner consisting of 5 GMPRs. We partition the start and end points mainly based on ***multiplicity***, the number of unique geodesics between the pair (this strategy has been used for upper bounds on GC(tetrahedron) and GC(cube)).

- E_5 is a discrete set containing all (p, q) where p and q are octahedron vertices, or the midpoint of an edge or face (this contains all (p, q) with multiplicity 6, and there exist no points with multiplicity 5).
- E_1 (resp. E_4) contains all points not in E_5 with multiplicity 1 (resp. 4).
- E_3 contains all points not in E_5 with multiplicity 3 along with some with multiplicity 2 (these were added to ensure continuity of σ_2).
- E_2 contains the remaining points with multiplicity 2.

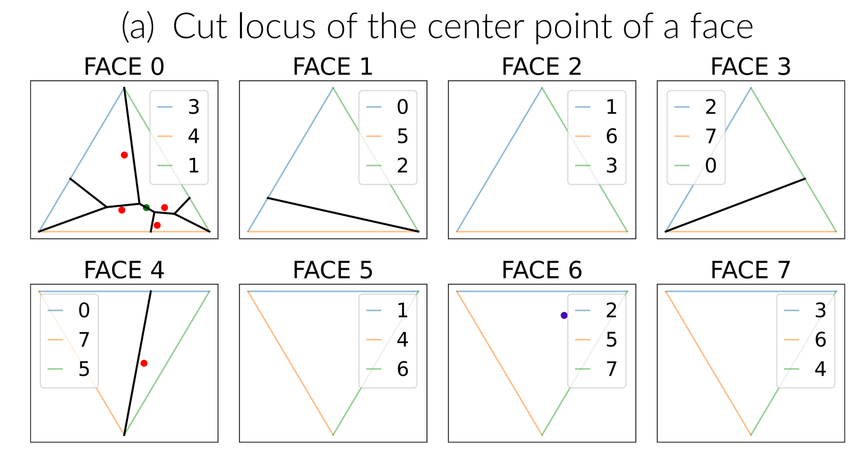
For most of these sets, the associated σ_i is easy to construct. For σ_1, there is only one choice (since any $(p, q) \in E_1$ has a unique geodesic). E_3, E_4, and E_5 can be partitioned into small clopen sets, so σ_3, σ_4, and σ_5 are easily constructed as piecewise functions. The most difficult GMPR is σ_2. To construct $\sigma_2(p, q)$, we consider the cut locus line that q lies on and approach from a counterclockwise direction, when applicable.



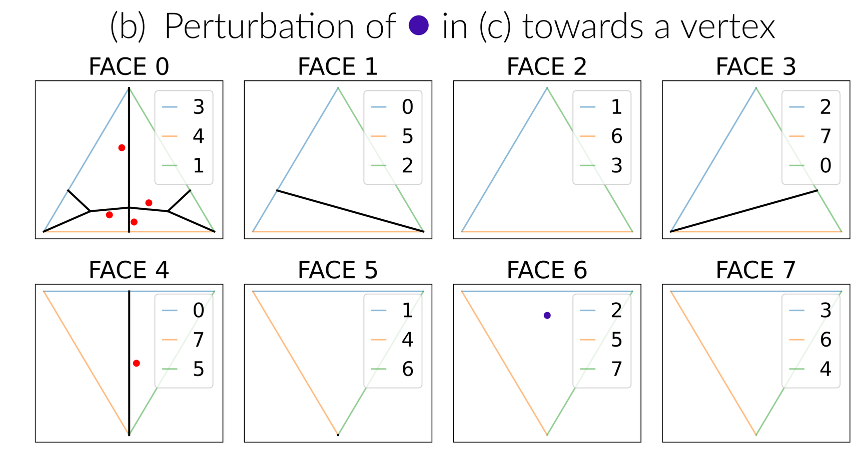
(a) Cut locus of the center point of a face



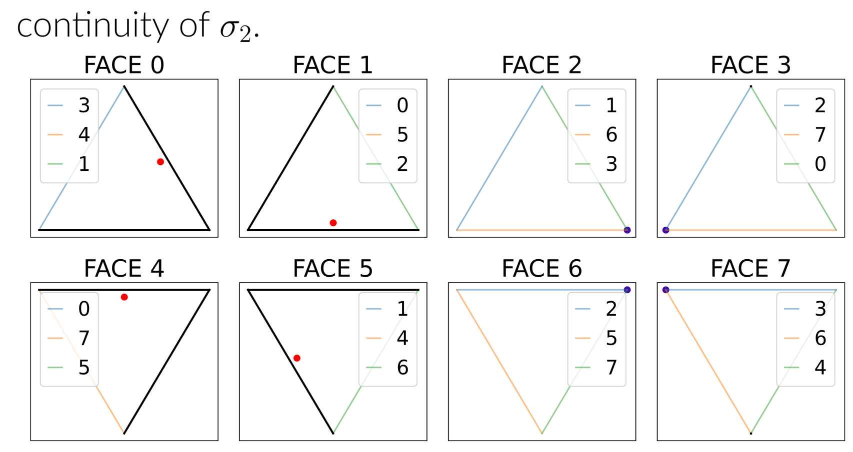
(b) Perturbation of • in (c) towards a vertex



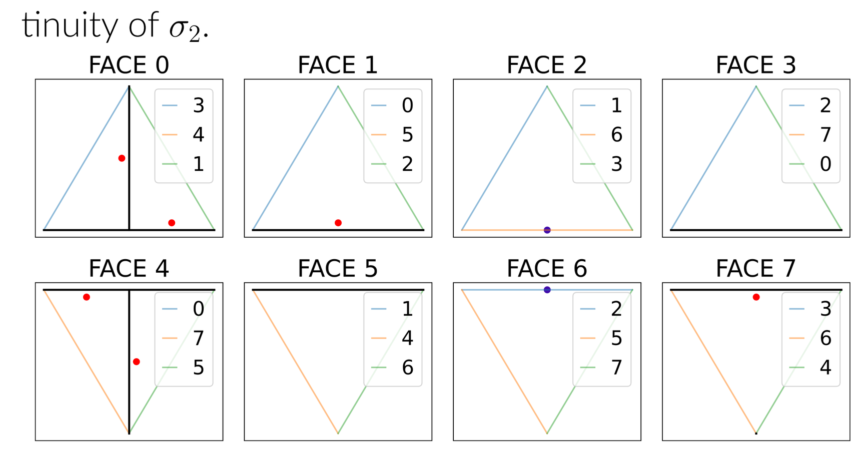
(c) Further perturbation of • in (d) towards an edge. The cut locus lines with no specified • are in E_3 to preserve continuity of σ_2.



(d) Perturbation of • in (c) towards an edge. The cut locus lines with no specified • are in E_3 to preserve continuity of σ_2.



(e) Cut locus of a vertex



(f) Cut locus of a point on an edge

Figure 3. Construction of GMPR σ_2. A path starting from • approaches a cut locus line from the side indicated by •. • indicates free choice on which side to approach from. Observe that when a cut locus line has a unique endpoint on an octahedron vertex, σ_2 approaches the line from the 'right' side of that vertex (i.e. the first side encountered when moving counterclockwise around that vertex).

Lower bound on GC(octahedron)

To show $GC(\text{octahedron}) \geq 4$, we use our simplex embedding theorem on embeddings around a particular point of the cut locus. We construct collections of $(0, \ldots, 4)$-dimensional simplex embeddings to show this lower bound (this is the the strongest bound possible with our method). The cut loci relevant to our embeddings are those in Figure 3(a,b,c).

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