A New Ribbon Basis for Rank-Selected Homology of Geometric Lattices & Representation Stability

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Plan for Today

I. Background

II. Main Results

- construct basis for rank-selected homology of any geometric lattice P (called "ribbon basis") (positively answers 1982 question of Anders Björner)

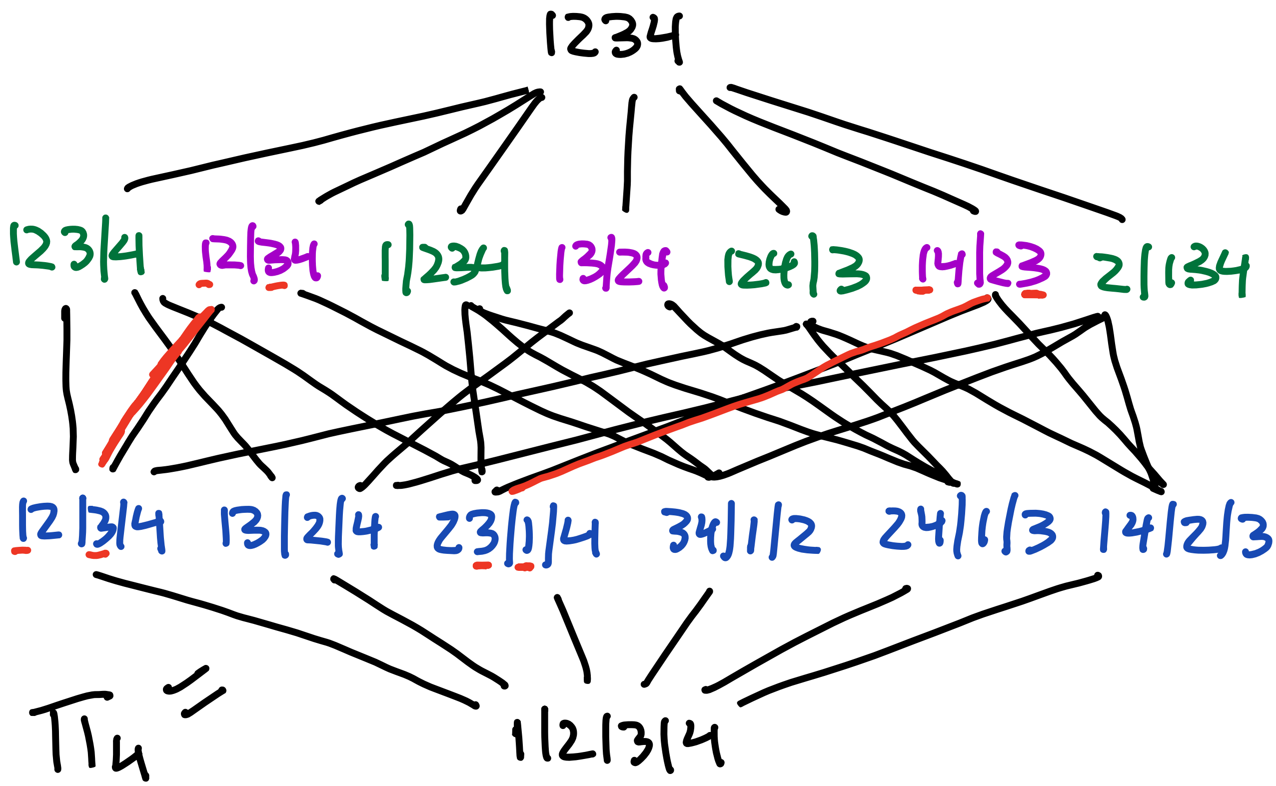
- (1) establish representation stability of $\beta_S(\Pi_n)$ for partition lattice Π_n
  (2) prove sharp stability bound of $4\max S - |S| + 1$ for $\beta_S(\Pi_n)$ (proving conjecture of H.-Reiner)

- establish stability & prove sharp stability bound for Boolean lattice B_n

III. Examples

Partition Lattice Π_n and its S_n-Representations

$\Pi_4 =$



- Π_n poset of set partitions w/ refinement order
- S_n acts on Π_n by permuting values

  e.g. $(13)[12|3|45] = 32|1|45 \in \Pi_5$

- Action preserves $<$, so permutes chains $u_1 < u_2 < \ldots < u_k$, i.e., faces of $\Delta(\Pi_n)$

S_n-action on chains of P gives rise to S_n-representations on Homology Groups of $\Delta(P)$

- S_n-action on Π_n also preserves poset rank (# steps from $\hat{0}$). S_n-action on $B_n :=$ poset of subsets of $\{1,2,\ldots,n\}$ with $\subseteq$ order preserves $<$ and rank too
- S_n-action on chains of Π_n (resp. B_n) commutes with simplicial boundary map d given by:

$$d(u_0 < \ldots < u_r) = \sum_{i=0}^{r} (-1)^i (u_0 < \ldots < \hat{u_i} < \ldots < u_r)$$

for $\Delta(\Pi_n)$ (resp. $\Delta(B_n)$), hence induces S_n-representation on i^{th} homology group of order complex $\Delta(P)$ for each i.

Def'n: Given graded poset P of rank n with G-action preserving rank & $<$, define for each $S \subseteq \{1,2,\ldots,n-1\}$:
$$S = \{s_1 < s_2 < \cdots < s_k\}$$

$\alpha_S(P) :=$ permutation rep'n of G on chains $c_1 < \cdots < c_k$ in P where $\mathrm{rank}(c_i) = s_i$ for $i=1,2,\ldots,k$

$$\beta_S(P) := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(P)$$ = "virtual" G-rep'n

Fact: If graded P with G-action is "Cohen-Macaulay" (e.g., shellable), then:

(a) $P^S = \{u \in P \mid rk(u) \in S\}$ has homology concentrated in top degree $\forall\, S \subseteq \{1,\ldots,rkP-1\}$

& (b) $\beta_S(P)$ is an actual G-representation, namely rep'n on top homology of P^S coming from G-action on P^S.

Rank-Selected Homology $\beta_S(\Pi_n)$ of Partition Lattice Π_n

$S = \{1,2,\ldots,n-2\}$ case:

(where $\Pi_n^{\{1,\ldots,n-2\}} = \Pi_n \setminus \{\hat{0},\hat{1}\}$ since $\mathrm{rank}(\Pi_n) = n-1$)

Thm (Hanlon + Stanley):

$$\beta_{\{1,\ldots,n-2\}}(\Pi_n) \cong_{S_n} \mathrm{sgn}_n \otimes \left(\varphi \uparrow_{C_n}^{S_n}\right)$$

where φ is linear rep'n sending generator for C_n to $e^{2\pi i/n}$

Rk: $\dim(\beta_{\{1,\ldots,n-2\}}(\Pi_n)) = (n-1)! = \pm \mu_{\Pi_n}(\hat{0},\hat{1})$

Thm (Joyal): $\beta_{1,\ldots,n-2}(\Pi_n) \cong_{S_n} \mathrm{sgn}_n \otimes \underbrace{\mathrm{Lie}_n}_{\text{multilinear part free Lie alg.}}$

Qn (Richard Stanley, '82): Understand $\beta_S(\Pi_n)$ for each $S \subseteq \{1, \ldots, n-2\}$.

- Lots of past work on $\langle \beta_S(\Pi_n), \mathbb{1} \rangle$, but even that not well understood for $S \neq \{1, 2, , n-2\}$

  (encapsulates famous hard questions regarding plethysm of rep'ns)

Today: "representation stability" for $\beta_S(\Pi_n)$ for fixed S (as n grows) & sharp representation stability bound

What rep. stability bound gets you:

- for fixed S, knowing $\beta_S(\Pi_n)$ for $n = 1, 2, 3, \ldots, \underbrace{4 \max S - |S| + 1}_{\text{stability bound we prove}}$ determines $\beta_S(\Pi_n)$ for all n.

e.g. $S = \{1, 3\} \Rightarrow 4 \max S - |S| + 1 = 12 - 2 + 1 = \boxed{11}$

Rk: Rank-selection is poset analogue of partial flag variety (clearest example: poset of subspaces)

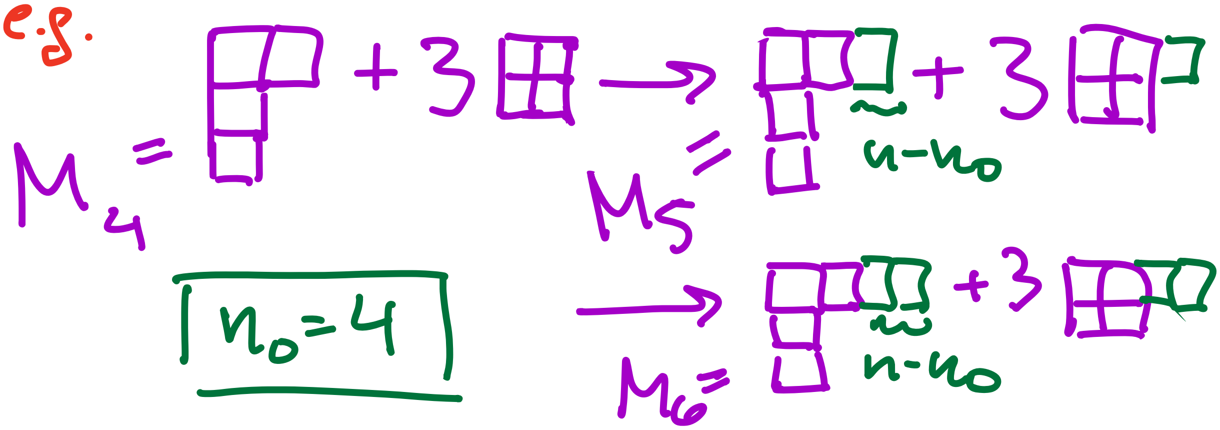
Representation Theoretic Stability

Def'n (Church-Farb): A series of S_n-modules $M_1, M_2, M_3, \ldots$ for $n=1,2,3,\ldots$ stabilizes at $n=n_0$ if

$$M_{n_0}=\bigoplus_{\lambda\vdash n_0} c_\lambda S^\lambda \Rightarrow M_n=\bigoplus_{\lambda\vdash n_0} c_\lambda S^{(\lambda_1+n-n_0,\lambda_2,\lambda_3,\ldots)}$$

(for all $n\geq n_0$)

e.g.



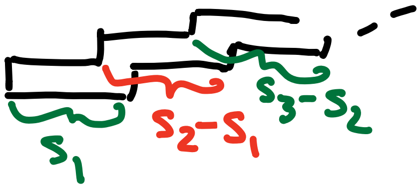
Stability Proof Method: Decompose each M_n into components $S^\lambda\otimes \mathbb{1}_{n-|\lambda|}\uparrow_{S_{|\lambda|}\times S_{n-|\lambda|}}^{S_n}$ and give upper bound on $|\lambda|+\lambda_1$.

S_n- Module Structure for $\beta_S(B_n)$

Thm (Solomon): $\beta_S(B_n) \cong_{S_n} S$ 

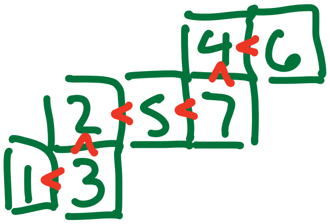
Specht module of ribbon shape

for $S=\{s_1,\ldots,s_k\}$
with $1 \le s_1 < \ldots < s_k \le n-1$



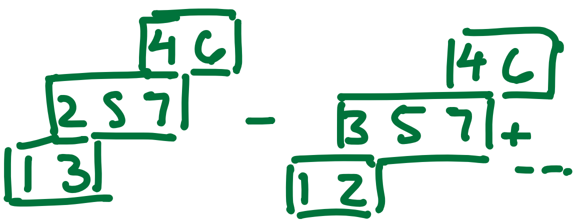
Plan: Homology basis for $\beta_S(P)$ for geometric lattices P sharing features of "polytabloid" basis for Specht modules useful for calculating irrep. mults

e.g. (Boolean lattice case)

$F =$ 

F = standard Young tableau

$\rightsquigarrow$ v_F = polytabloid basis element =



ribbon basis element

$\rightsquigarrow$ $(\{1,3\} < \{1,3,2,5,7\} < \ldots \quad)$
$-(\{1,2\} < \{1,2,3,5,7\} < \ldots \quad) + \ldots$

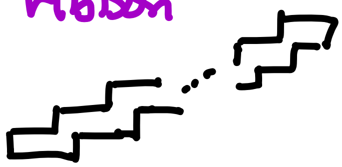
II. Main Results

1. Fix finite $S \subseteq \{1,2,\ldots\}$. Letting n grow, we prove:
   $\{1,2,\ldots\} = \mathbb{Z}_{>0}$

   - $\beta_S(B_n)$ is representation stable.
   - it has sharp uniform representation stability bound $2\max S - |S| + 1$

2. We construct new "ribbon basis" for $\beta_S(P)$ for any geometric lattice P

   - answers question of Anders Björner
   - introduces matroid/graphical analogue of Specht modules (for ribbon shapes)



3. Fixing $S \subseteq \mathbb{Z}_{>0}$ and letting n grow,

we prove:

- representation stability for $\beta_S(\Pi_n)$
- $\beta_S(\Pi_n)$ satisfies sharp (uniform) representation stability bd of $4 \max S - |S| + 1$ (as conjectured by H.-Reiner)

Note: special cases of $S = \{1, 2, \ldots, i\}$ & $S = \{i\}$ previously done in H-Reiner

Motivations: computer pfs, understand multiplicities..

Techniques:

- symmetric fns & Whitney homology (for Boolean lattices)
- Young symmetrizers applied to "ribbon basis" in rather nonstandard way (for partition lattices)

III. Examples

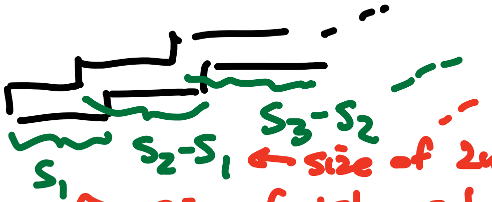
Homology Basis Construction for $\beta_S(\Pi_n)$

- totally order atoms $a_1, a_2, \ldots, a_n$ (elements covering $\hat{0}$)

  e.g. $12 <_{\pi} 13 <_{\pi} 23 <_{\pi} 14 <_{\pi} 24 <_{\pi} 34 <_{\pi} \cdots$ in Π_n

  where $12 := 12|3|4|\cdots|n$ $\quad ij := ij|1|2|\cdots|\hat{i}|\cdots|\hat{j}|\cdots$
  $13 := 13|2|4|5|\cdots|n$

- consider "standard" fillings of ribbon shape



for $S = \{s_1 < s_2 < \cdots\}$

$\leftarrow$ size of 2nd poset rank jump in P^S

$\leftarrow$ size of 1st poset rank jump in P^S

with atoms (increase L to R in rows & decrease up columns)

e.g.

$$\begin{array}{ccccc} & 13 & 58 & 67 & \cdots \\ 12 & 14 & & & \end{array}$$

$a_{i_3} < a_{i_4} < a_{i_5}$

$\wedge$

$a_{i_1} < a_{i_2}$

Such that

(1) $a_{i_1} <\cdot\ a_{i_1} \vee a_{i_2} <\cdot\ a_{i_1} \vee a_{i_2} \vee a_{i_3} <\cdot\ \cdots$

in geometric lattice P

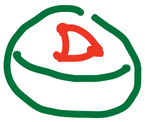
(2) each a_{i_j} small as possible w.r.t. Γ yielding poset chain $a_{i_1} \lessdot a_{i_1} \vee a_{i_2} \lessdot \cdots$

e.g. $12 \vee 14 = 12 \vee 24$ but $14 <_P 24$

so $\cancel{\boxed{12}\boxed{24}}$ $\quad \boxed{12}\boxed{14}$ ✓

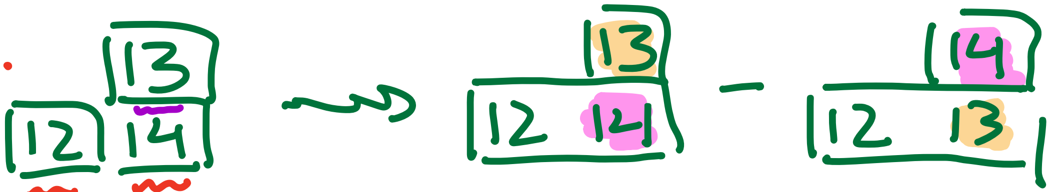
- allowable fillings map bijectively to the "homology facets" in shelling for $\Delta(P^S)$

facets "closing off" spheres



- send each such ribbon filling F to "polytabloid" v_F

e.g.



- map v_F to alternating sum of maximal chains in P^S giving ribbon basis element indexed by F.

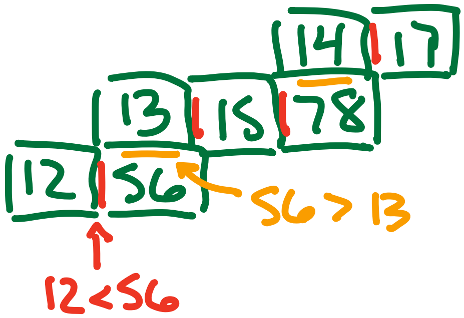
e.g. $(12 \vee 14 <_P 12 \vee 14 \vee 13) - (12 \vee 13 <_P 12 \vee 13 \vee 14)$

More Detailed Example of Ribbon Basis Element

e.g. $n=8$, $S=\{2,5\}$ $\rightsquigarrow$ ribbon row lengths $2, 5-2, 8-5$

Total order on atoms of Π_n:

$$\underset{\substack{\updownarrow \\ 12|3|4|\cdots|8}}{12} < 13 < 23 < 14 < 24 < 34 < 15 < 25 < \cdots$$

$F =$ 

= "standard" Young tableau whose entries are NBC^+ basis

$$\bar{f}_{chain}(\{F\}) = \underbrace{\underset{\substack{\shortparallel \\ 12|56|3|4|7|8}}{e_{12} \vee e_{56}} < \underset{\substack{\shortparallel \\ 12356|78|4}}{e_{12} \vee e_{56} \vee e_{13} \vee e_{15} \vee e_{78}}}_{\text{max chain in } \Pi_n^S \text{ for } S=\{2,5\}}$$

map sending tabloids to max chains in P^S

Basis Element indexed by F

$$F = \begin{array}{ccccc} & & & 14 & 17 \\ & 13 & 15 & 78 & \\ 12 & 56 & & & \end{array}$$

$$\bar{f}_{chain}(v_F) = \bar{f}_{chain}\left(\begin{array}{ccccc} & & & 14 & 17 \\ & 13 & 15 & 78 & \\ 12 & 56 & & & \end{array} - \begin{array}{ccccc} & & & 78 & 17 \\ & 13 & 15 & 14 & \\ 12 & 56 & & & \end{array} - \begin{array}{ccccc} & & & 14 & 17 \\ & 56 & 15 & 78 & \\ 12 & 13 & & & \end{array} + \begin{array}{ccccc} & & & 78 & 17 \\ & 56 & 15 & 14 & \\ 12 & 13 & & & \end{array} \right)$$

$$= (12 \vee 56 < (12 \vee 56) \vee (13 \vee 15 \vee 78))$$
$$- (12 \vee 56 < (12 \vee 56) \vee (13 \vee 15 \vee 14))$$
$$- (12 \vee 13 < (12 \vee 13) \vee (56 \vee 15 \vee 78))$$
$$+ (12 \vee 13 < (12 \vee 13) \vee (56 \vee 15 \vee 14))$$

Example: Stability Bound for $B_S(\Pi_n)$

- Prove $|\lambda| + \lambda_1 > 4\max S - |S| + 1 \implies \langle \underbrace{\widehat{WH}_S}_{\text{related to } B_S}(\Pi_n), S^\lambda \rangle = 0$

to deduce stability bd $4\max S - |S| + 1$

e.g. $S = \{1,3\}$, $\lambda = (6)$ $T = \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}$

$\max S = 3$ $|S| = 2$

$|\lambda| = 6$
$\lambda_1 = 6$
shape $= (6)$

$$\underset{\shortparallel}{4\max S - |S| + 1} \qquad \underset{\shortparallel}{|\lambda| + \lambda_1}$$

$$4\cdot\underbrace{3}_{\max S} - \underbrace{2}_{|S|} + 1 = 11 < 12 = 6 + 6$$

$$b_T a_T = \underbrace{(1)}_{\text{column antisymmetrizer}} \Big(\underbrace{\sum_{\sigma \in S_6}}_{\text{row symmetrizer}} \sigma \Big)$$

"Young symmetrizer" $= b_T a_T$

$$\underbrace{\bar{f}_{chain}(v_F)}_{\text{ribbon basis elt}} = \bar{f}_{chain}\left(\begin{smallmatrix} 12 & 56 \\ 34 & \end{smallmatrix} - \begin{smallmatrix} 34 & 56 \\ 12 & \end{smallmatrix} \right)$$

$$= \boxed{(1|2|34|5|6 < 12|34|56) - (12|3|4|5|6 < 12|34|56)}$$

$$\underbrace{b_T a_T(\sim)}_{\text{apply to ribbon basis elt}} = \bar{f}_{chain}\Big(\underbrace{\left(\begin{smallmatrix} 12 & 56 \\ 34 & \end{smallmatrix} - \begin{smallmatrix} 34 & 56 \\ 12 & \end{smallmatrix} \right)}_{v_F} + \underbrace{(13)(24)}_{\in S_6}(v_F) \pm \cdots \Big)$$

$$= \bar{f}_{chain}\left[\left(\overbrace{\cancel{\begin{smallmatrix}1&2&5&6\\3&4&&\end{smallmatrix}}}^{F} - \cancel{\begin{smallmatrix}3&4&5&6\\1&2&&\end{smallmatrix}}\right) + \left(\overbrace{\cancel{\begin{smallmatrix}3&4&5&6\\1&2&&\end{smallmatrix}}}^{(13)(24)F} - \cancel{\begin{smallmatrix}1&2&5&6\\3&4&&\end{smallmatrix}}\right) + \cancel{\ldots}\ \ \cancel{\ldots}\ \ \cdots\right]$$

$= 0$ (since summands cancel in pairs)

One may show:

$b_T a_T \bar{f}_{chain}(v_F) = 0$ $\forall$ T of shape $\square\square\square\square$ & all $\bar{f}_{chain}(v_F)$ in ribbon basis

so $b_T a_T(\widehat{WH}_{\{1,3\}}(\Pi_n)) = 0$ (since $b_T a_T$ sends basis for $\beta_S(\Pi_n)$ to 0)

so $\langle \widehat{WH}_{\{1,3\}}(\Pi_n), S^{\square\square\square\square} \rangle = 0$ (argument works for $|\lambda| + \lambda_1 > 4\max S - |S| + 1$)

implying stability for $n \geq 4\max S - |S| + 1$.

Thank you!