Totally Nonnegative Matrices and Zeros of Poset-polynomials

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based on joint work with
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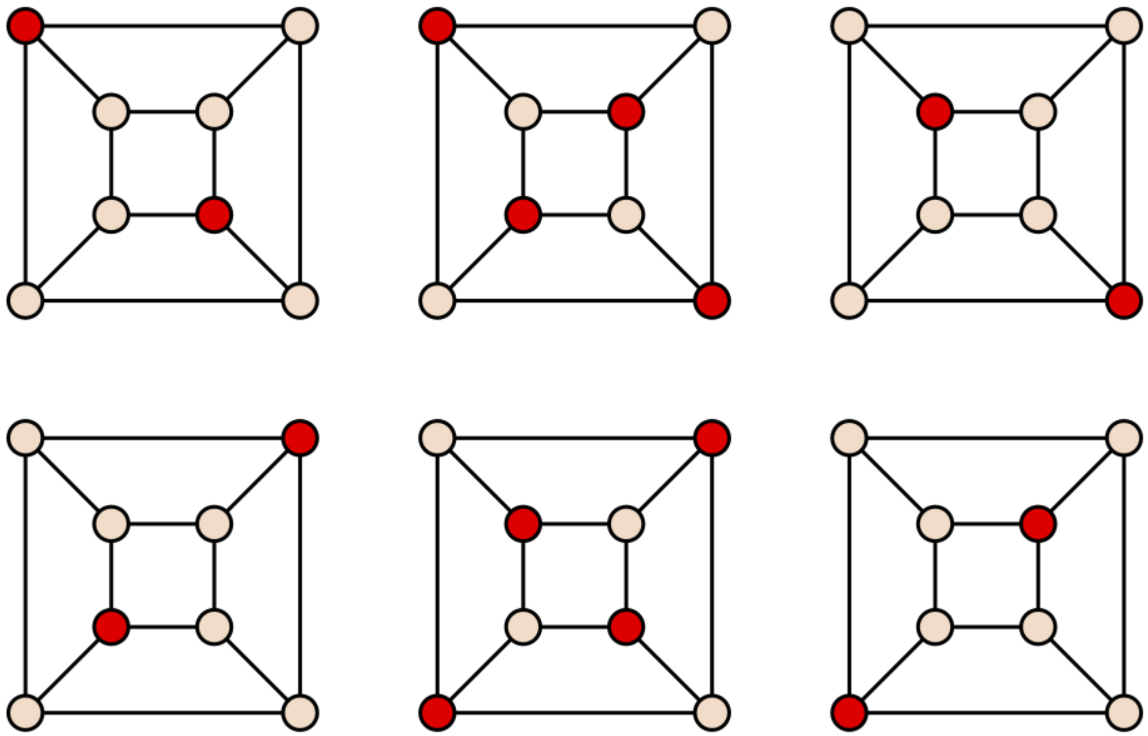
Plan

- Negative Dependence and Independence Polynomials
- Chain Polynomials of Posets and Matrices
- Using Totally Nonnegative Matrices
- Chow Polynomials

Topic 1

Negative Dependence and Independence Polynomials

- Let $G = (V, E)$ be a graph.
- A set S of vertices is called independent if it does not contain both vertices of an edge.



- Let $G = (V, E)$ be a finite graph.
- A set S of vertices is independent if it does not contain both vertices of an edge.
- The independence polynomial or the partition function of the Hard-core repulsive lattice-gas model (HRM):

$$I_G(t) = \sum_{S \text{ independent}} t^{|S|} = \sum_{k=0}^{|V|} i_k(G) \cdot t^k.$$

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- Since HRM models repelling particles one would expect negative dependence properties of $I_G(t)$?

Univariate negative dependence inequalities.

- Suppose μ is a probability measure on subsets of a finite set V, modeling sets of occupied sites of identical particles.

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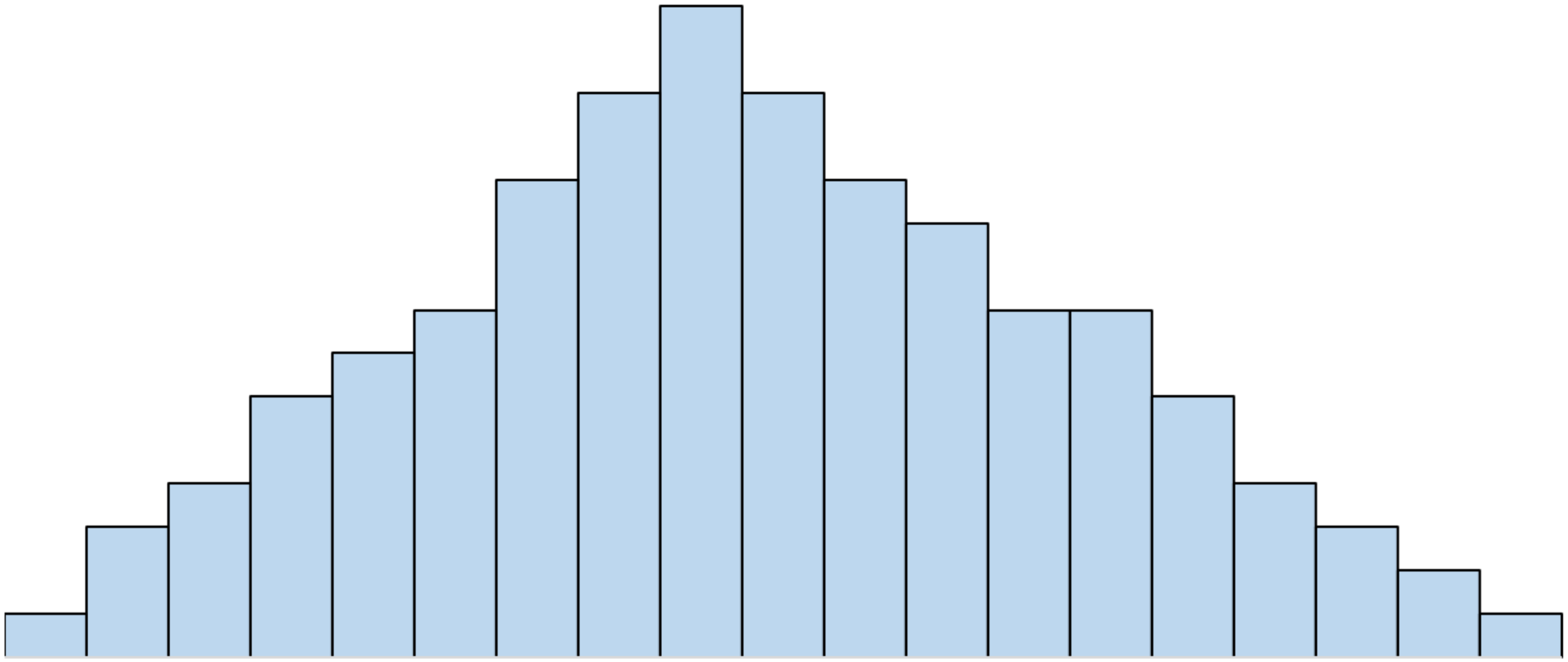
- Unimodality: $r_0 \leq r_1 \leq \cdots \leq r_m \geq r_{m+1} \geq r_{m+2} \geq \cdots$.



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- Log-concavity: $r_k^2 \geq r_{k-1} r_{k+1}, \quad \text{for } 0 < k < n = |V|$.
- The polynomial $r_0 + r_1 t + r_2 t^2 + \cdots + r_n t^n$ is real-rooted, i.e., all of its zeros are real.

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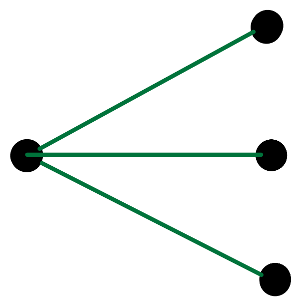
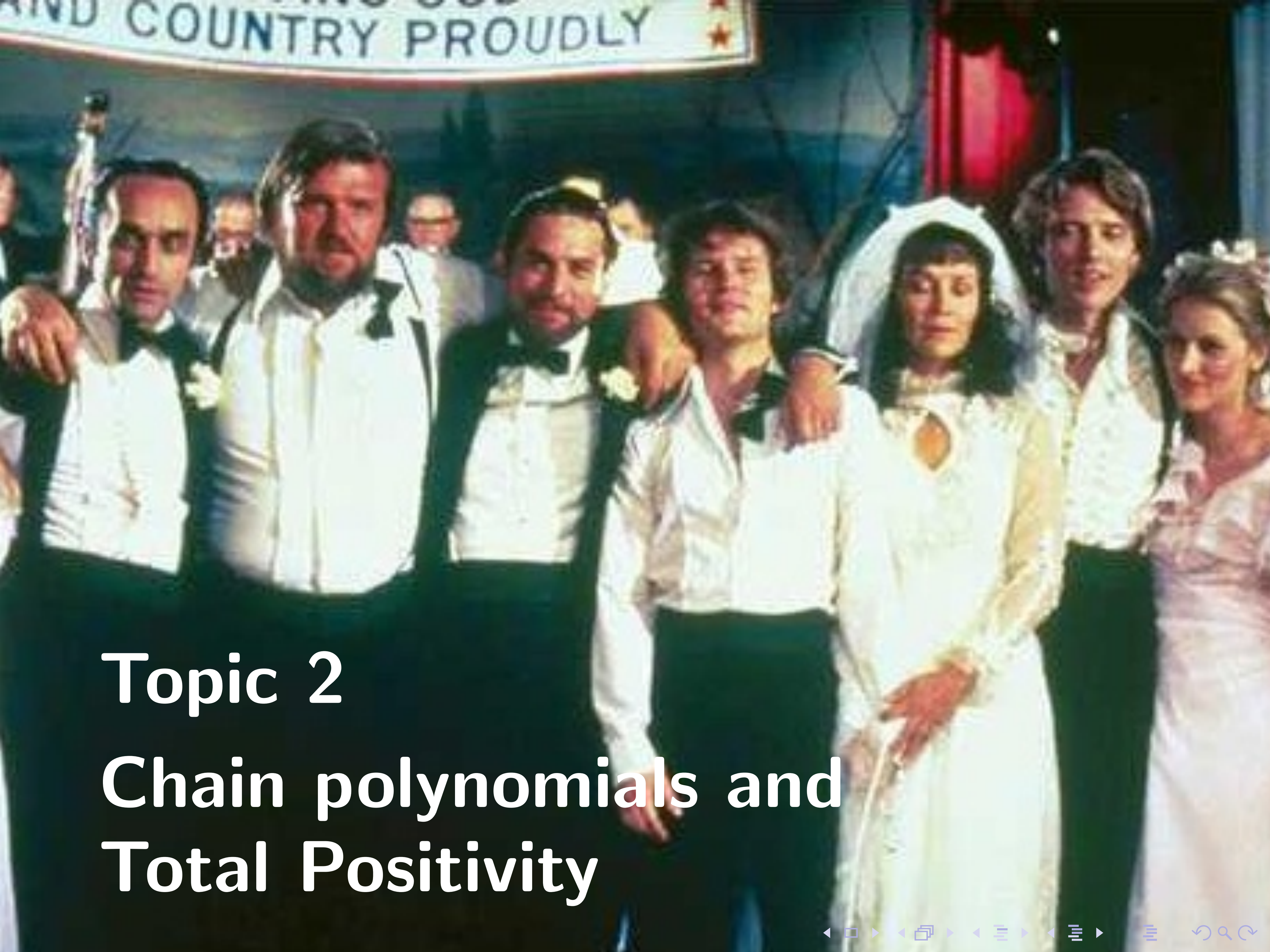
- Theorem (Chudnovsky–Seymour, 2007). The independence polynomial of a claw-free graph is real-rooted.

No induced claw

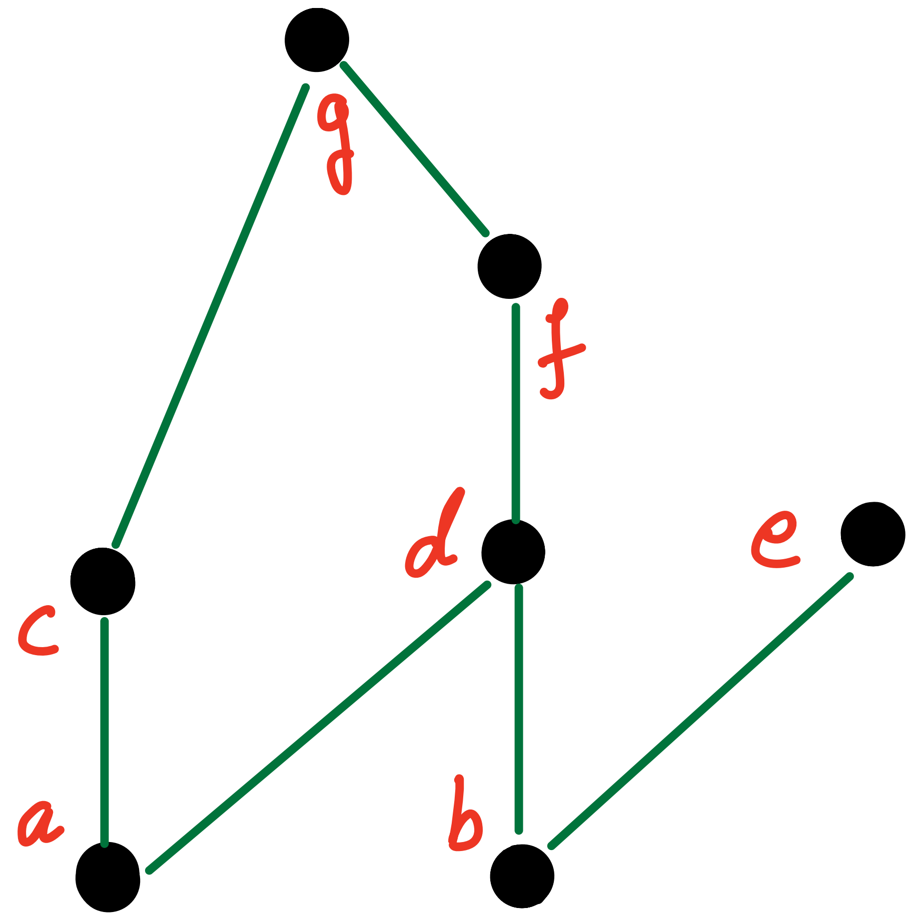




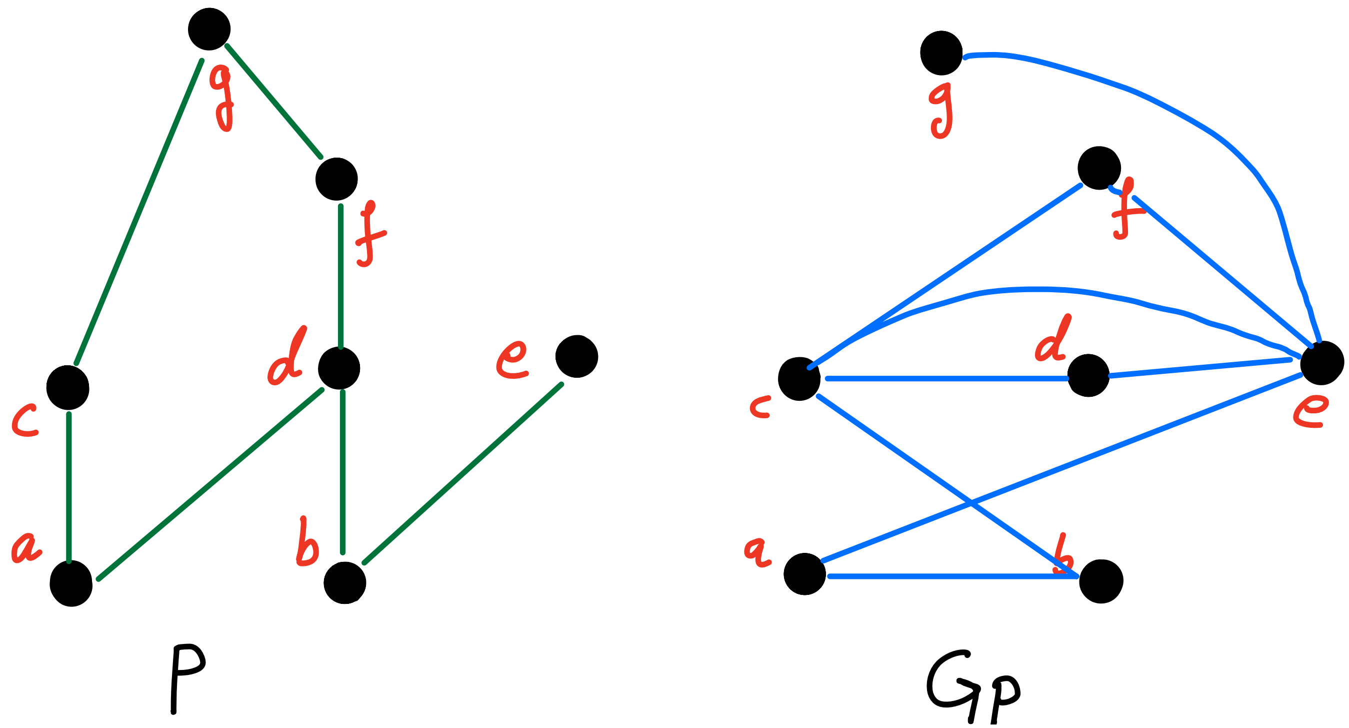
Topic 2

Chain polynomials and Total Positivity

- Let $P = (V, \leq)$ be a finite partially ordered set (poset).



- Let $P = (V, \leq)$ be a finite partially ordered set (poset).
- The anti-adjacency graph, G_P, of a poset P is the graph on V where $\{x, y\}$ is an edge iff x and y are not related in P.



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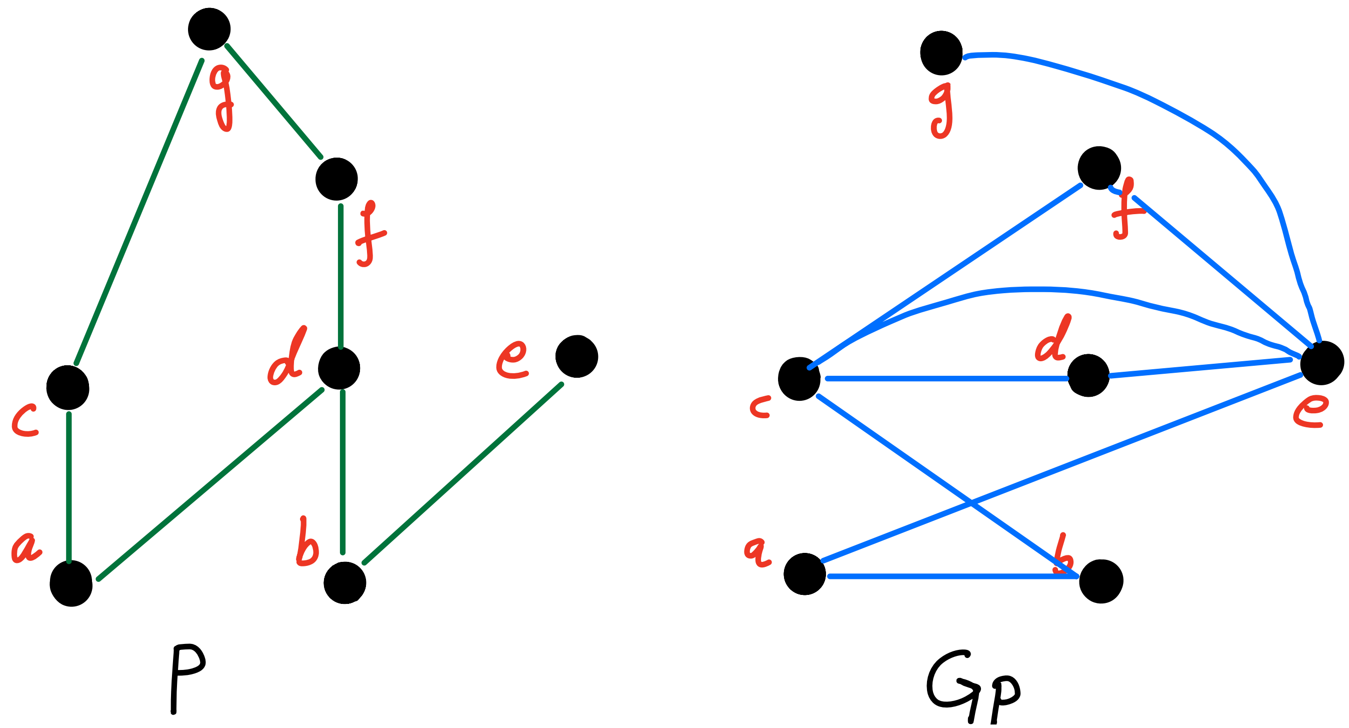
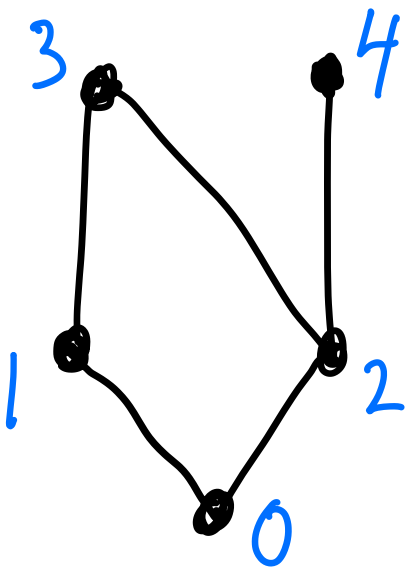


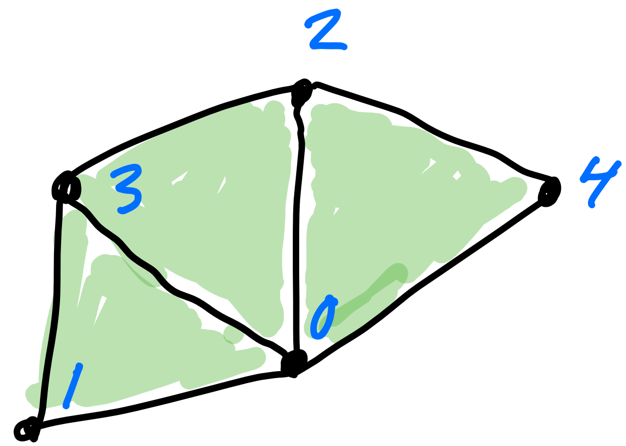
- $S \subseteq V$ is independent in G_P iff S is a chain in P, i.e., a totally ordered subset of V.

The chain polynomial of P is the independence polynomial of G_P:

$$c_P(t) = \sum_{k=0}^{|V|} c_k(P) \cdot t^k,$$

where $c_k(P)$ is the number of chains of size k in P.





- $c_P(t) = 1 + 5t + 7t^2 + 3t^3 = (1 + 3t)(1 + t)(1 + t)$.

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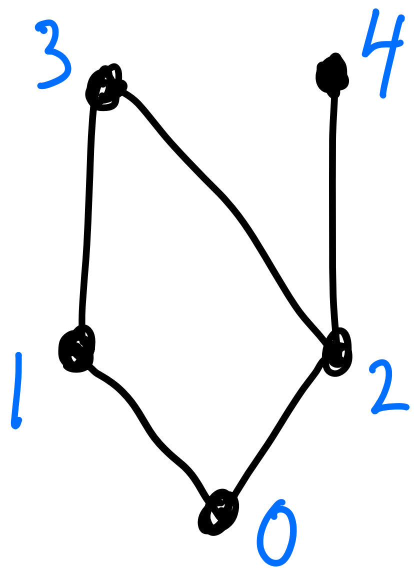

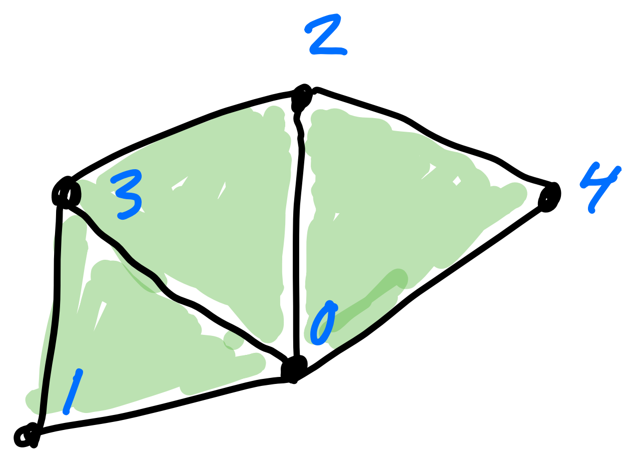

- $c_P(t) = 1 + 5t + 7t^2 + 3t^3 = (1+3t)(1+t)(1+t)$.
- Hence the chain polynomials is the f-polynomial of the order complex of P.

- A matrix $R = (r_{n,k})_{n,k=0}^{N}$ is unitriangular if it is lower triangular and all diagonal elements are equal to 1.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ r_{1,0} & 1 & 0 & 0 & 0 & 0 & \cdots \\ r_{2,0} & r_{2,1} & 1 & 0 & 0 & 0 & \cdots \\ r_{3,0} & r_{3,1} & r_{3,2} & 1 & 0 & 0 & \cdots \\ r_{4,0} & r_{4,1} & r_{4,2} & r_{4,3} & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- A matrix $R = (r_{n,k})_{n,k=0}^{N}$ is unitriangular if it is lower triangular and all diagonal elements are equal to 1.
- The chain polynomials associated to R are (I = identity)

$$p_n(t) = \left[(I - t(R - I))^{-1}\right]_{n,0} = \sum_{k=0}^{n} t^k \left[(R - I)^k\right]_{n,0}.$$

- Equivalently, $p_0(t) = 1$, and

$$p_n(t) = t \sum_{k=0}^{n-1} r_{n,k} \cdot p_k(t), \qquad n \geq 1.$$

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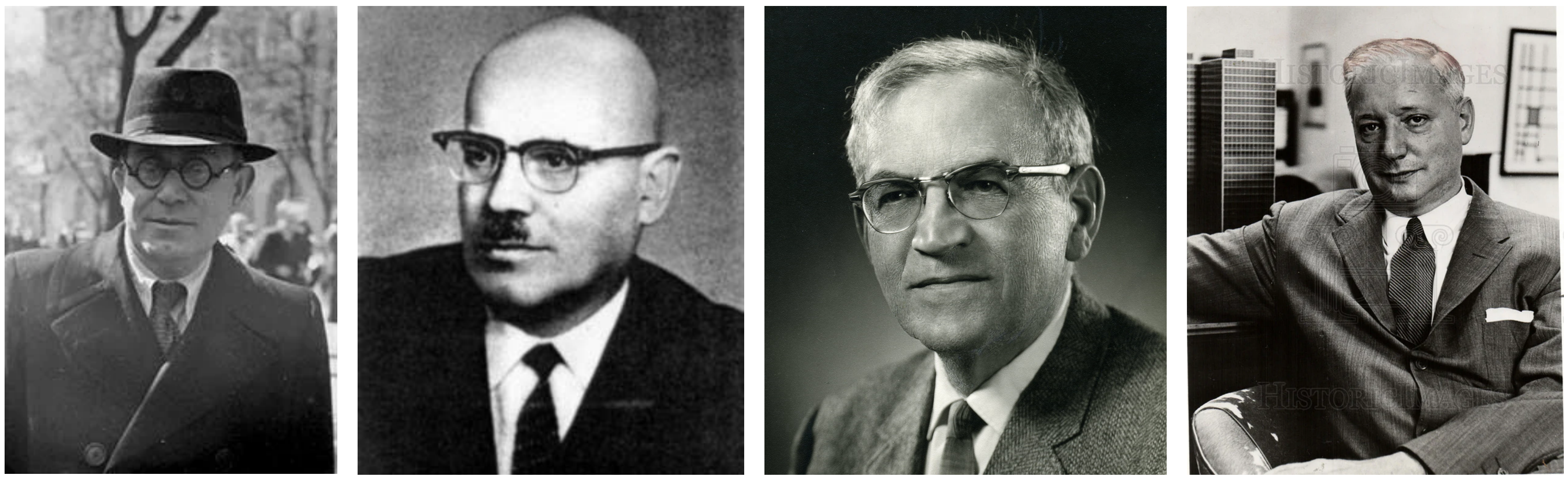
- Example. If $r_{n,k} \equiv 1$, then $p_n(t) = t(1+t)^{n-1}$.
- Example. If $r_{n,k} \equiv \binom{n}{k}$, then

$$p_n(t) = \sum_{k=1}^{n} k! \left\{ {n \atop k} \right\} t^k,$$

where $\left\{ {n \atop k} \right\}$ is a Stirling number of the second kind.

- A matrix with real entries is totally nonnegative (TN) if every minor is nonnegative.
- Important in linear algebra, statistics, analysis, geometry, combinatorics and physics.



Felix Gantmacher, Mark Krein, Isaac Schoenberg, Samuel Karlin

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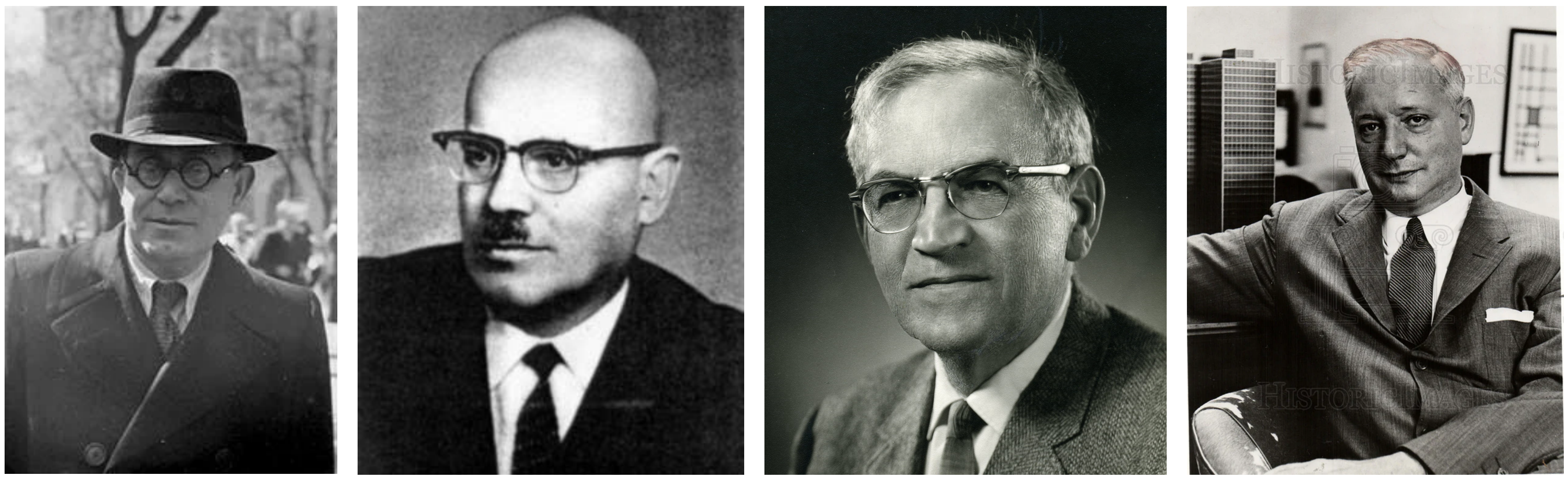

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- Theorem (B., Saud, 2026). If R is unitriangular and TN, then all zeros of $p_n(t)$ are real and located in the interval $[-1, 0]$.

- $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ is a Pólya frequency sequence (PF) if the Toeplitz matrix $(a_{i-j})_{i,j=0}^{\infty}$ is TN, where $a_k = 0$ if $k < 0$.

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots \\ a_4 & a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- Theorem (Aissen, Schoenberg, Whitney, Edrei, Thoma). $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ is PF iff

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = C z^N e^{\gamma z} \prod_{i=1}^{\infty} \frac{1 + \alpha_i z}{1 - \beta_i z},$$

where $C, \gamma, \alpha_i, \beta_i$ are nonnegative real numbers, $N \in \mathbb{N}$, and $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) < \infty$.

- Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{R}[[z]]$ be a formal power series, and consider

$$\frac{1}{1 - t(f(z) - f(0))} = \sum_{n=0}^{\infty} f_n(t) z^n \in \mathbb{R}[[z, t]].$$

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- Equivalently,

$$f_n(t)=\sum_{k=0}^{n} t^k \sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{Z}_{>0}^k\\ n_1+\cdots+n_k=n}} a_{n_1}a_{n_2}\cdots a_{n_k}.$$

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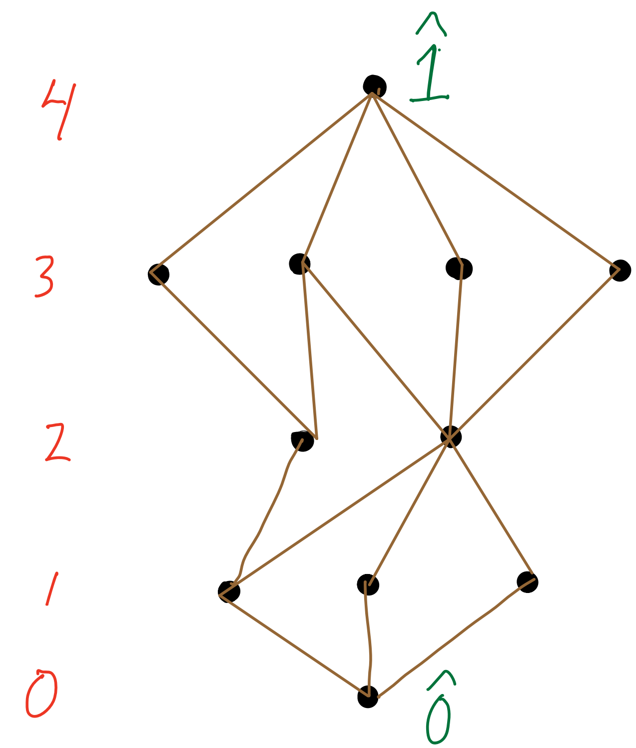
- Theorem (B., Saud, 2026). If $\{a_n\}_{n=0}^{\infty}$ is PF, then $f_n(t)$ is real-rooted.
- A special case was conjectured by Forgács and Tran (2016).

Poset terminology

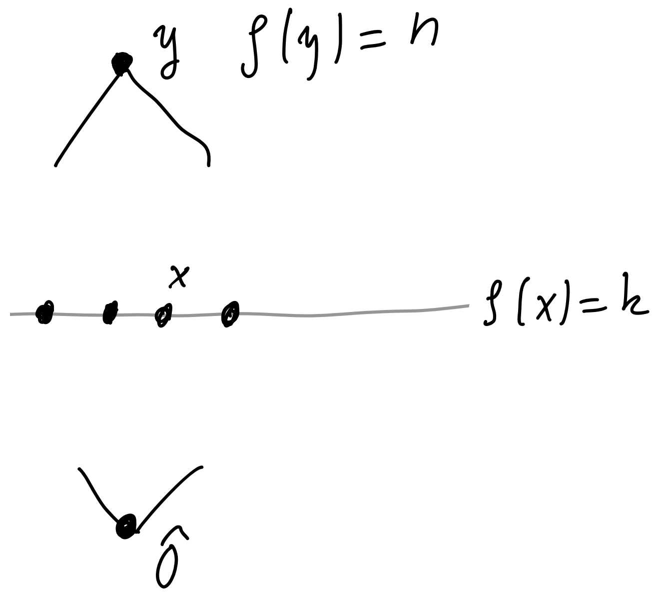
- Let P be a partially ordered set (poset).
- P has a least element $\hat{0}$ if $\hat{0} \leq x$ for all $x \in P$.
- P has a greatest element $\hat{1}$ if $x \leq \hat{1}$ for all $x \in P$.
- A poset P with $\hat{0}$ is graded if there exists a rank function $\rho : P \to \mathbb{N}$ s.t.
  - $\rho(\hat{0}) = 0$,
  - $\rho(y) = \rho(x) + 1$ if y covers x.



- An interval in a poset P is a subposet of the form

$$[x, y] = \{z \in P : x \leq z \leq y\}, \qquad \text{where } x \leq y \in P.$$

- A graded poset P with $\hat{0}$ is called uniform if there is a matrix $R(P) = (r_{n,k}(P))_{n,k=0}^{r}$ such that
- there are exactly $r_{n,k}(P)$ elements of rank k below each element of rank n.



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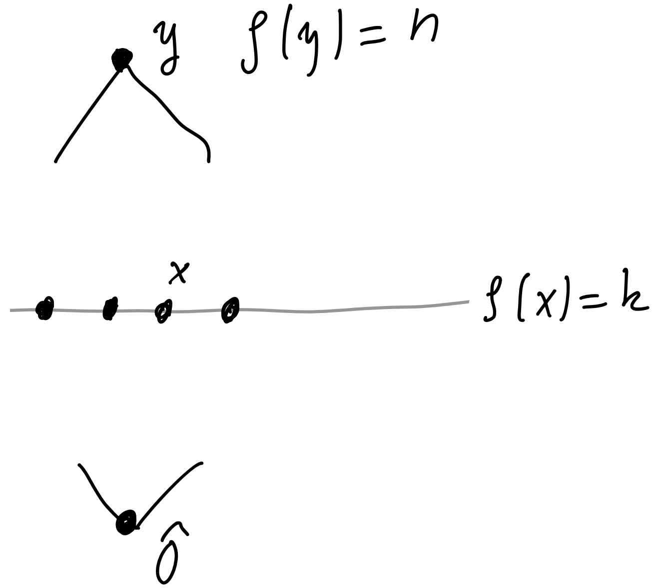

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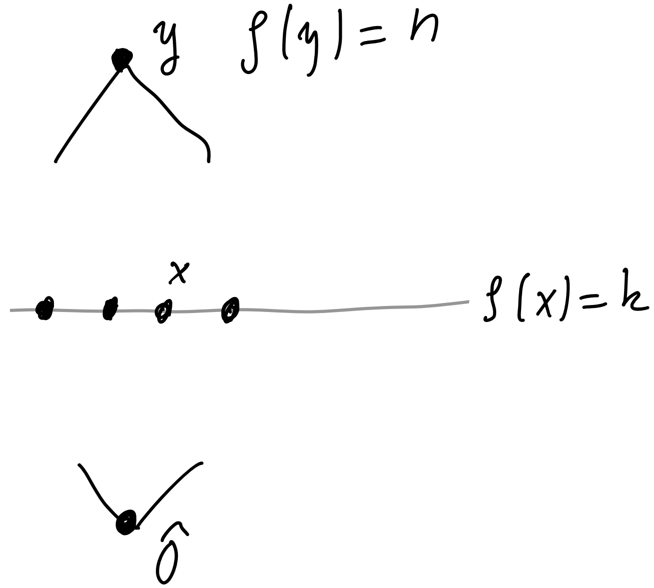

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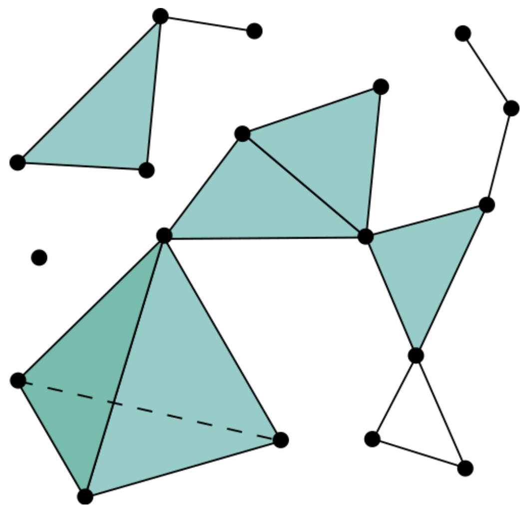
- $R(P)$ is unitriangular.
- For example binomial posets and Sheffer posets are uniform.

Example: Simplicial complexes

- Recall that an (abstract) simplicial complex on a finite vertex set V is a collection Δ of subsets of V, for which

(1) $\{v\} \in \Delta$ for each $v \in V$, and

(2) if $\sigma \subseteq \tau$ and $\tau \in \Delta$, then $\sigma \in \Delta$.



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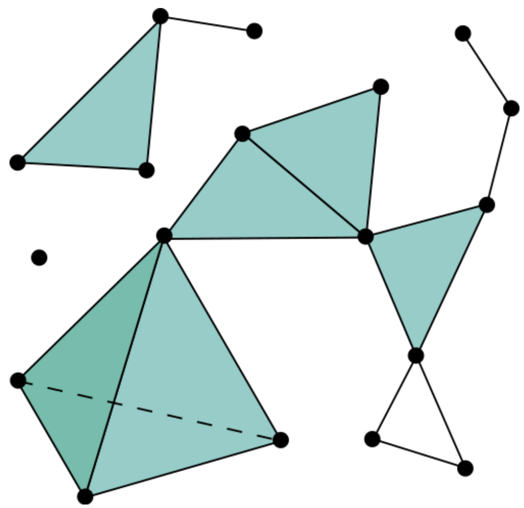

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- Each interval $[\varnothing, \sigma]$ is isomorphic to the boolean algebra $\mathbb{B}_m$ of rank $m = \#\sigma$.

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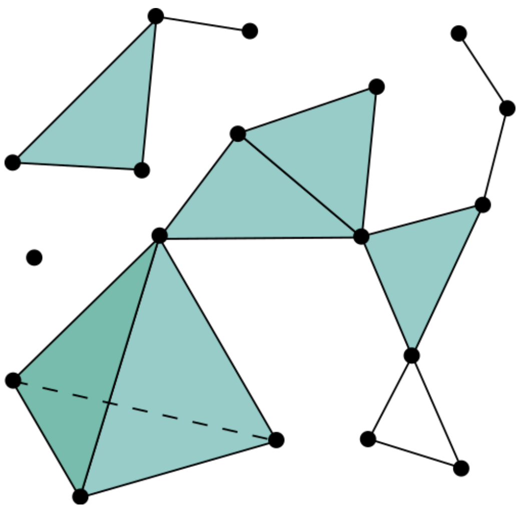

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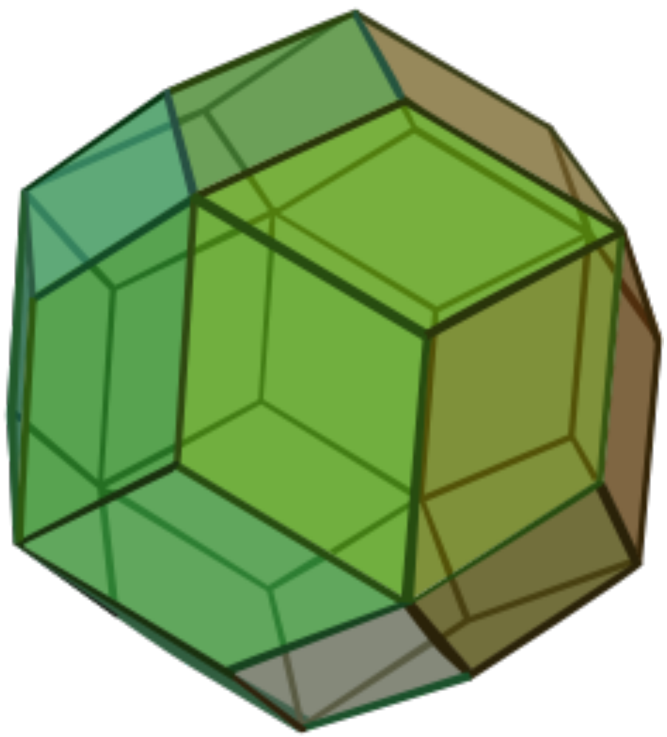
- Each interval $[\varnothing, \sigma]$ is isomorphic to the boolean algebra $\mathbb{B}_m$ of rank $m = \#\sigma$.
- $r_{n,k}(\Delta) = \binom{n}{k}$.

Example: Cubical complexes

- A cubical complex is a poset for which each interval $[\hat{0}, x]$ is isomorphic to the face lattice of a cube of the appropriate dimension.
- For example face lattices of cubical polytopes.



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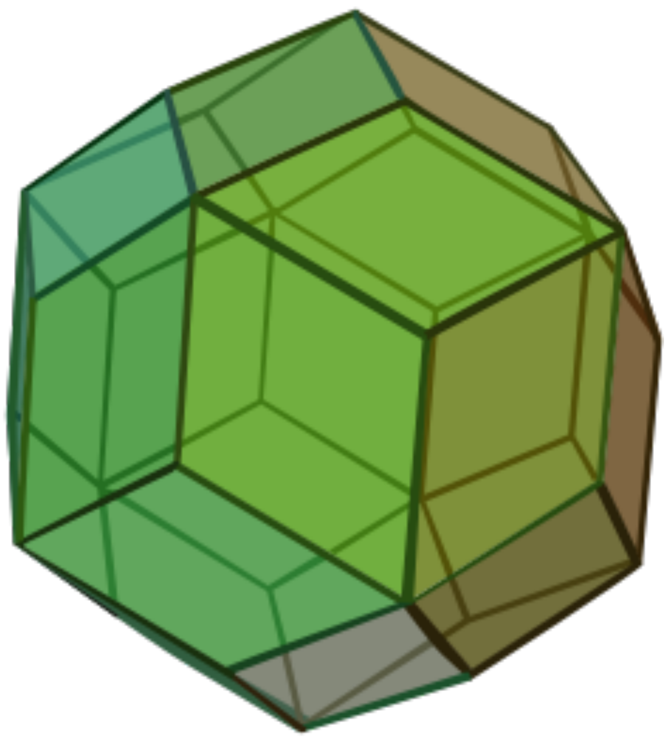

- $r_{n,k}(P) = 2^{n-k}\binom{n-1}{k-1}$, for $k \geq 1$.

Example: Projective geometries and q-complexes

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- Let $\mathbb{B}_n(q)$ be the projective geometry of all subspaces of $\mathbb{F}_q^n$.

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- $r_{n,k}(P) = \binom{n}{k}_q$, a Gaussian binomial coefficient.

- If P is uniform and $R = R(P)$, then

$$c_{[\hat{0},x]}(t) = \frac{1}{t}(1+t)^2 p_n(t), \quad \text{whenever } \rho(x) = n > 1.$$

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- If P is uniform and $R = R(P)$, then

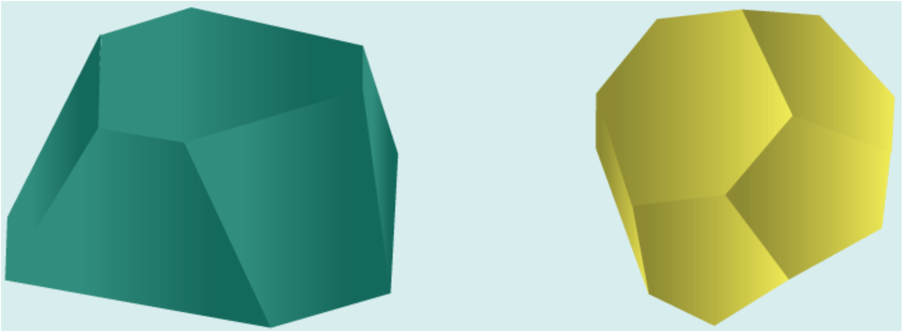
$$c_{[\hat{0},x]}(t) = \frac{1}{t}(1+t)^2 p_n(t), \quad \text{whenever } \rho(x) = n > 1.$$

- Hence the chain polynomials of $R(P)$ are essentially the chain polynomials of the lower intervals of P.
- A uniform poset P is called TN if $R(P)$ is TN.
- Corollary (B., Saud, 2026). The chain polynomial of a TN-poset with $\hat{1}$ is real-rooted.

Examples of TN-posets

- A poset P is pure if all maximal elements have the same rank.
- $\Delta \cup \{\hat{1}\}$, where Δ is a pure simplicial complex with nonnegative h-vector.
- $\Delta \cup \{\hat{1}\}$, where Δ is a pure cubical complex with nonnegative h-vector.
- $\Delta \cup \{\hat{1}\}$, where Δ is a pure q-complex with nonnegative h-vector.
- (rank-) uniform geometric lattices.
- Duals of partition lattices and Dowling lattices.

- The face poset $F(\mathcal{P})$ of a convex polytope $\mathcal{P}$ are the faces of $\mathcal{P}$ ordered by containment.



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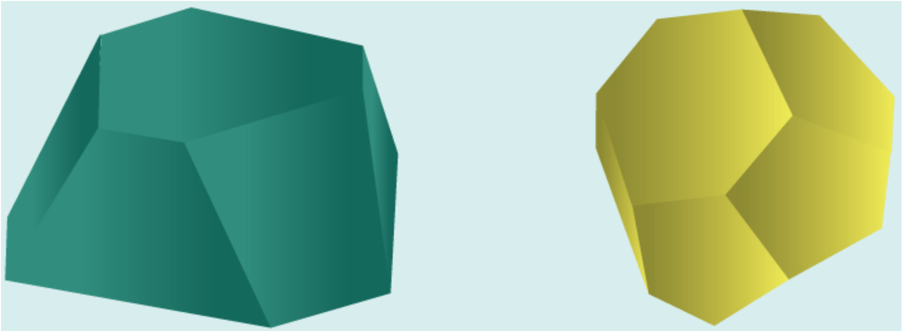

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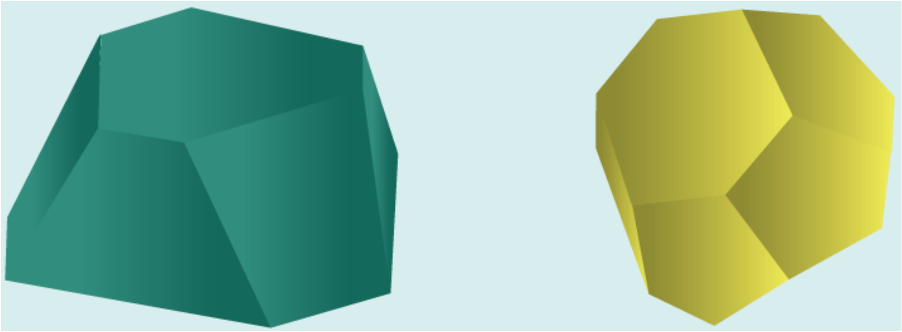

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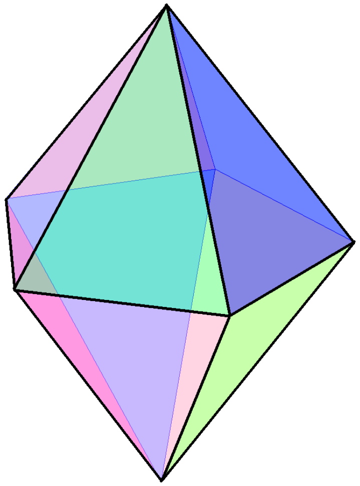

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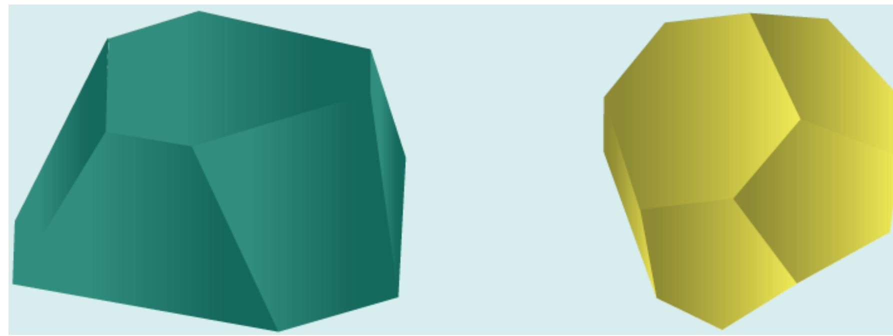

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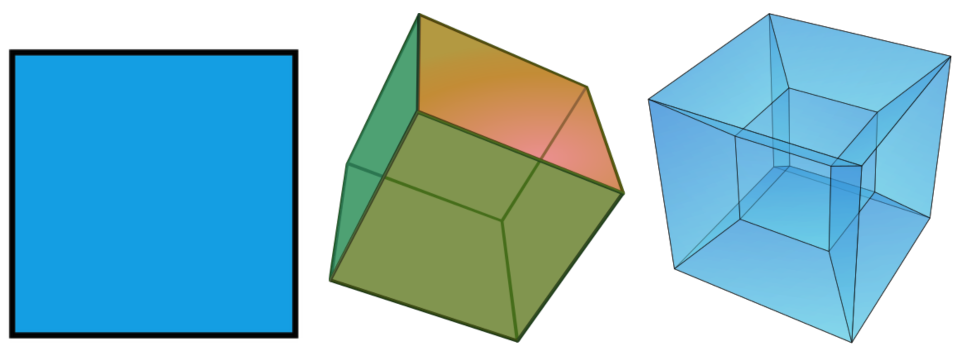
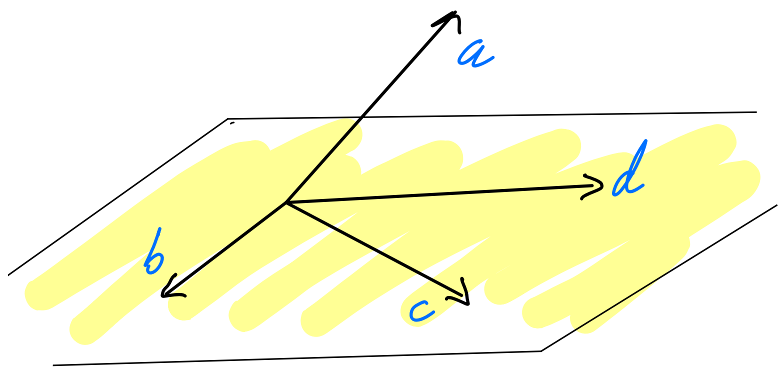
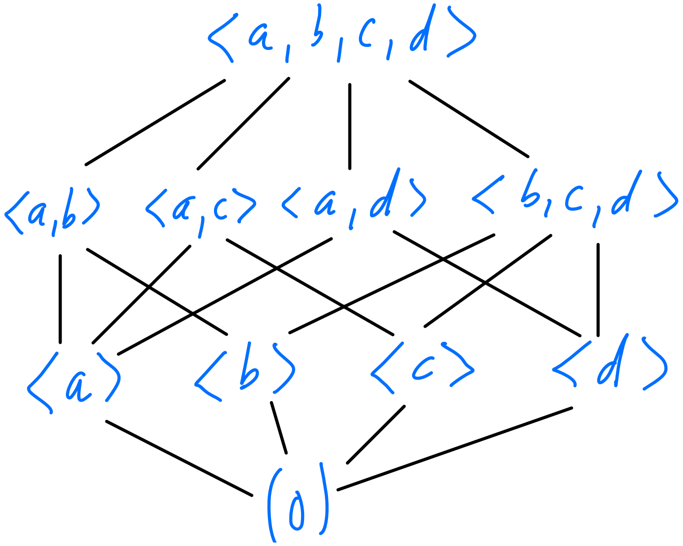
    - Theorem (Athanasiadis, 2021). True if $\mathcal{P}$ is cubical, i.e., all the faces are (hyper-)cubes.



- Let $V = \{v_1, \ldots, v_n\}$ be a set of vectors in a linear space $\mathcal{U}$.
- Let L be the poset of all subspaces of $\mathcal{U}$ that are generated by vectors in V.





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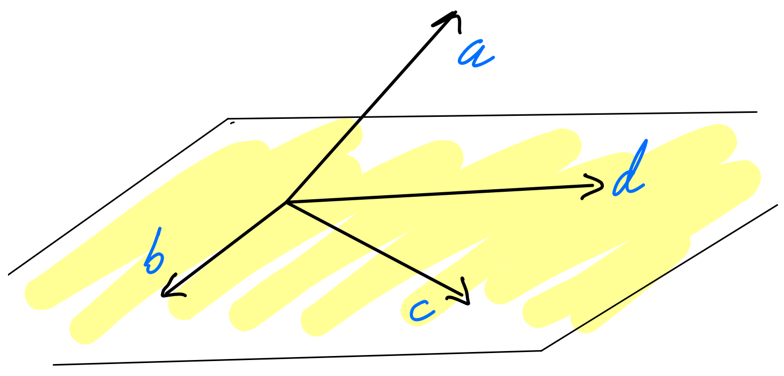

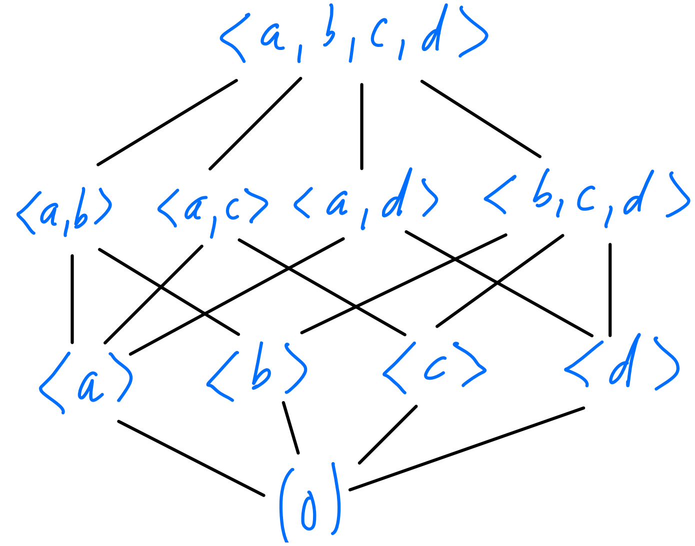

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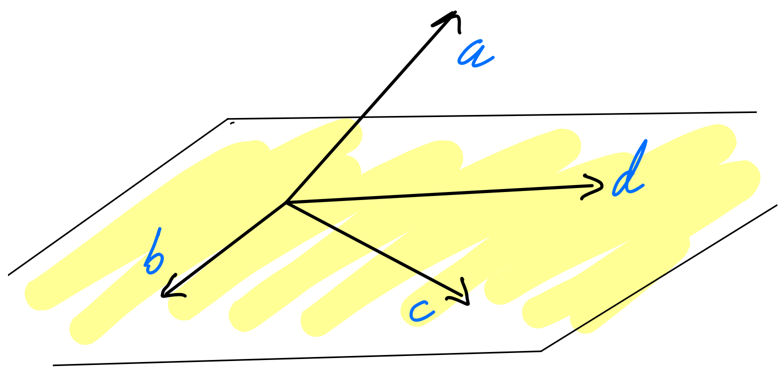

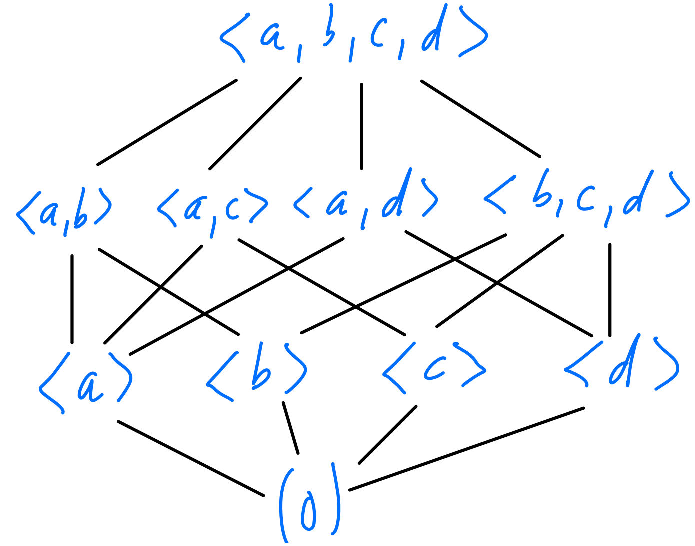

- L is a geometric lattice.
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- Theorem (B., Saud, 2026). True for (rank-) uniform geometric lattices, Dowling lattices, and paving lattices.

Main idea of proof

"Resolve (linearize) unitriangular TN-matrices"

Theorem (B., S., 26). Let $R = (r_{n,k})_{n,k=0}^{N}$ be unitriangular. TFAE

(TN) **TN**: R is totally nonnegative.

(RS) **Resolvable**: There are polynomials $R_{n,k}(t)$, $0 \le k \le n \le N$, and nonnegative real numbers $\lambda_{n,k}$ for which

- $R_{n,0}(t) = \sum_{k=0}^{n} r_{n,k} t^k$, "the row-generating polynomial"
- $R_{n+1,k}(t) = t^{n+1} + \sum_{j=k}^{n} \lambda_{n,j} \cdot R_{n,j}(t)$, for $0 \le k \le n+1$.

Proof sketch of "$p_n(t)$ is real-rooted"

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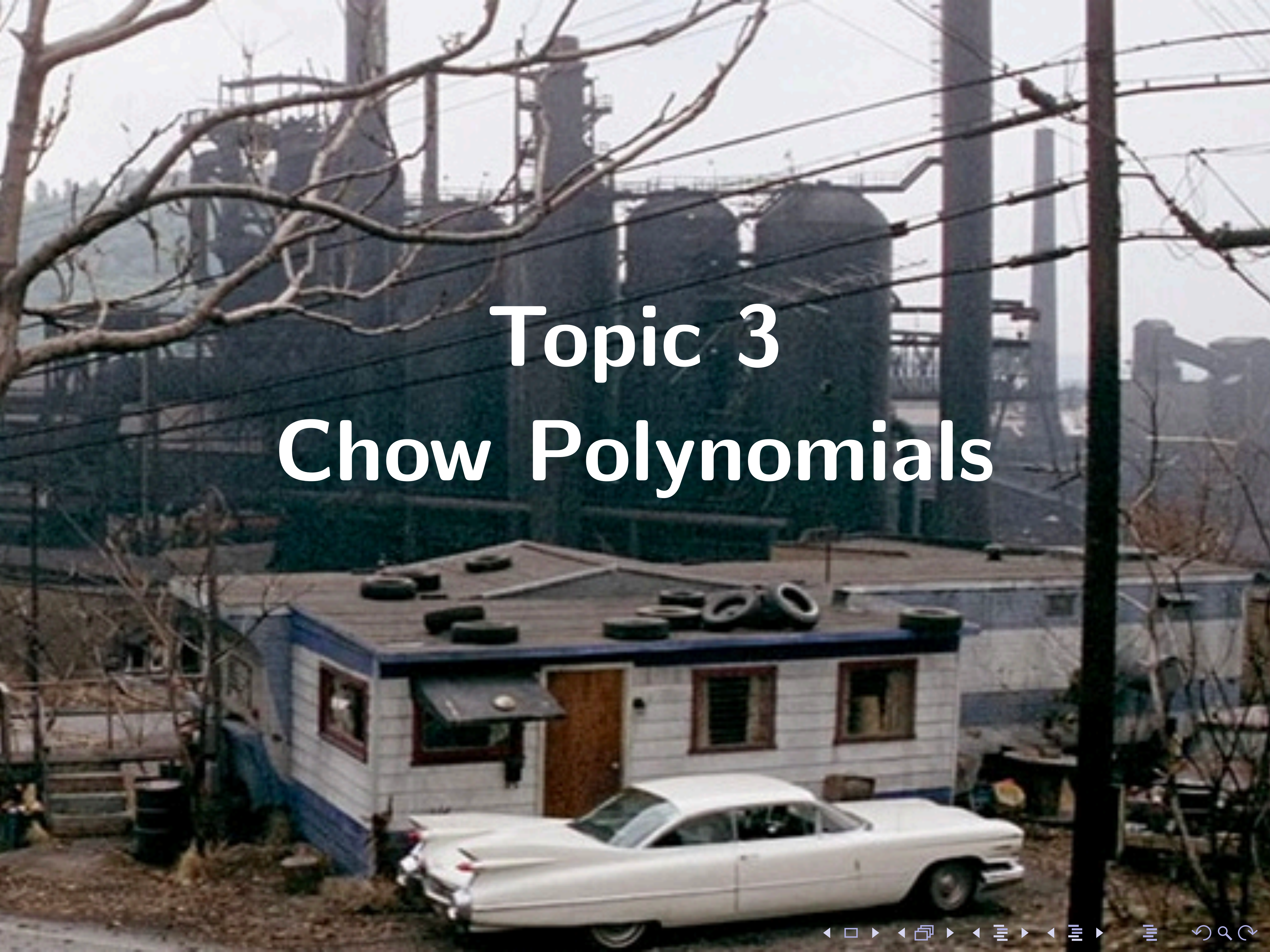
- By $(\star)$ and $(\star\star)$,

$$p_{n+1,k}(t) = t \sum_{j=0}^{k-1} \lambda_{n,j} \cdot p_{n,j}(t) + (1+t) \sum_{j=k}^{n} \lambda_{n,j} \cdot p_{n,j}(t)$$



Topic 3
Chow Polynomials

- The Eulerian polynomials, and derangement polynomials are

$$H_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} \quad \text{and} \quad d_n(t) = \sum_{\sigma \in \mathfrak{D}_n} t^{\mathrm{exc}(\sigma)}, \quad \text{where}$$

- $\mathfrak{S}_n$ is the symmetric group of permutations of $\{1, 2, \ldots, n\}$,
- $\mathfrak{D}_n$ is the set of derangements, i.e., fixed point free permutations, and
- $\mathrm{exc}(\sigma) = |\{i : \sigma(i) > i\}|$ is the number of excedances.

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- $H_n(t), d_n(t)$, $n \in \mathbb{N}$, satisfy

1. $H_0(t) = d_0(t) = 1$,
2. $t^{n-1} H_n(1/t) = H_n(t)$, for each $0 < n \leq N$,
3. $t^n d_n(1/t) = d_n(t)$, for each $0 \leq n \leq N$,
4. $H_n(t) = \sum_{k=0}^{n} \binom{n}{k} \cdot d_k(t)$, for each $0 \leq n \leq N$.

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4. $H_n(t) = \sum_{k=0}^{n} \binom{n}{k} \cdot d_k(t)$, for each $0 \leq n \leq N$.

- Properties 1–4 define $H_n(t)$ and $d_n(t)$ uniquely.

The Chow polynomial of a matrix

Theorem/Definition(B., Vecchi, 2025+). Let $R = (r_{n,k})_{n,k=0}^{N}$ be unitriangular. There are two unique sequences of polynomials $\{H_n(t)\}_{n=0}^{N}$ and $\{d_n(t)\}_{n=0}^{N}$ satisfying

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Moreover for $n \geq 1$,

$$d_n(t) = t\mathcal{S}_{n-1}\left(\sum_{k=0}^{n-1} r_{n,k} \cdot d_k(t)\right), \quad \text{where}$$

$$\mathcal{S}_n(f) = \frac{t^n f(1/t) - f(t)}{t-1}.$$

The augmented Chow polynomial of a matrix

Theorem/Definition (B., V., 2025+). Let $R = (r_{n,k})_{n,k=0}^{N}$ be unitriangular. There are two unique sequences of polynomials $\{G_n(t)\}_{n=0}^{N}$ and $\{A_n(t)\}_{n=0}^{N}$ satisfying

1. $A_0(t) = G_0(t) = 1$,
2. $t^n G_n(1/t) = G_n(t)$, for each $0 \leq n \leq N$,
3. $t^{n+1} A_n(1/t) = A_n(t)$, for each $0 < n \leq N$,
4. $G_n(t) = \sum_{k=0}^{n} r_{n,k} \cdot A_k(t)$, for each $0 \leq n \leq N$.

   Moreover, for $n \geq 1$,

$$A_n(t) = t\mathcal{S}_n\left(\sum_{k=0}^{n-1} r_{n,k} \cdot A_k(t)\right).$$

The augmented Chow polynomial of Pascal's triangle

- Let $R = (\binom{n}{k})_{n,k=0}^{N}$.
  1. $A_0(t) = G_0(t) = 1$,
  2. $t^n G_n(1/t) = G_n(t)$, for each $0 \leq n \leq N$,
  3. $t^{n+1} A_n(1/t) = A_n(t)$, for each $0 < n \leq N$,
  4. $G_n(t) = \sum_{k=0}^{n} \binom{n}{k} \cdot A_k(t)$, for each $0 \leq n \leq N$.
- These are satisfied by the Eulerian polynomials

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)+1}, \quad n > 0,$$

and the Binomial Eulerian polynomials

$$G_n(t) = \sum_{k=0}^{n} \binom{n}{k} \cdot A_k(t),$$

studied by Postnikov, Reiner, Williams, Shareshian, Wachs, Chung, Graham, Knuth, ...

- We call the polynomials associated to R:
  - $H_n(t)$ Chow polynomials
  - $d_n(t)$ Chow-derangement polynomials
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- If P is a uniform atomistic lattice, then the polynomials associated to $R(P)$ are Hilbert-Poincaré series of the (augmented) Chow ring of P defined by Feichtner-Yuzvinsky, 2004.

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- If P is a uniform atomistic lattice, then the polynomials associated to $R(P)$ are Hilbert-Poincaré series of the (augmented) Chow ring of P defined by Feichtner-Yuzvinsky, 2004.
- Conjecture (Huh & Stevens 2021, Ferroni & Schröter 2024). The Chow polynomials and augmented Chow polynomials of geometric lattices are real-rooted.

- Theorem (B., V., 2025+). If R is TN, then the polynomials $\{H_n(t)\}_{n=0}^N$, $\{d_n(t)\}_{n=0}^N$, $\{G_n(t)\}_{n=0}^N$ and $\{A_n(t)\}_{n=0}^N$ are real-rooted.

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- Theorem (B., V., 2025+). The (augmented) Chow polynomial of a TN-poset is real-rooted.
- Theorem (B., V., 2025+). The (augmented) Chow polynomial of a paving lattice is real-rooted.
- Theorem (Coron, Ferroni, Li, 2025+). The (augmented) Chow polynomial of UMEL-shellable posets are real-rooted.

- Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$, where $a_0 = 1$.
- Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{R}[[z]]$.
- Let $R = (a_{n-k})_{n,k=0}^{\infty}$, and consider the generating functions of the various Chow polynomials associated to R:

$$D(z,t) = \sum_{n=0}^{\infty} d_n(t) z^n, \qquad H(z,t) = \sum_{n=0}^{\infty} H_n(t) z^n,$$

$$A(z,t) = \sum_{n=0}^{\infty} A_n(t) z^n, \qquad G(z,t) = \sum_{n=0}^{\infty} G_n(t) z^n.$$

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- Corollary. For Pólya frequency sequences $\{a_n\}_{n=0}^{\infty}$, the polynomials $d_n(t), H_n(t), A_n(t), G_n(t)$ are real-rooted.

$$D(z,t) = \sum_{n=0}^{\infty} d_n(t) z^n = \frac{1-t}{f(tz) - tf(z)}$$

$$= \frac{1}{1 - \sum_{n\geq 2} a_n (t + t^2 + \cdots + t^{n-1}) z^n}$$

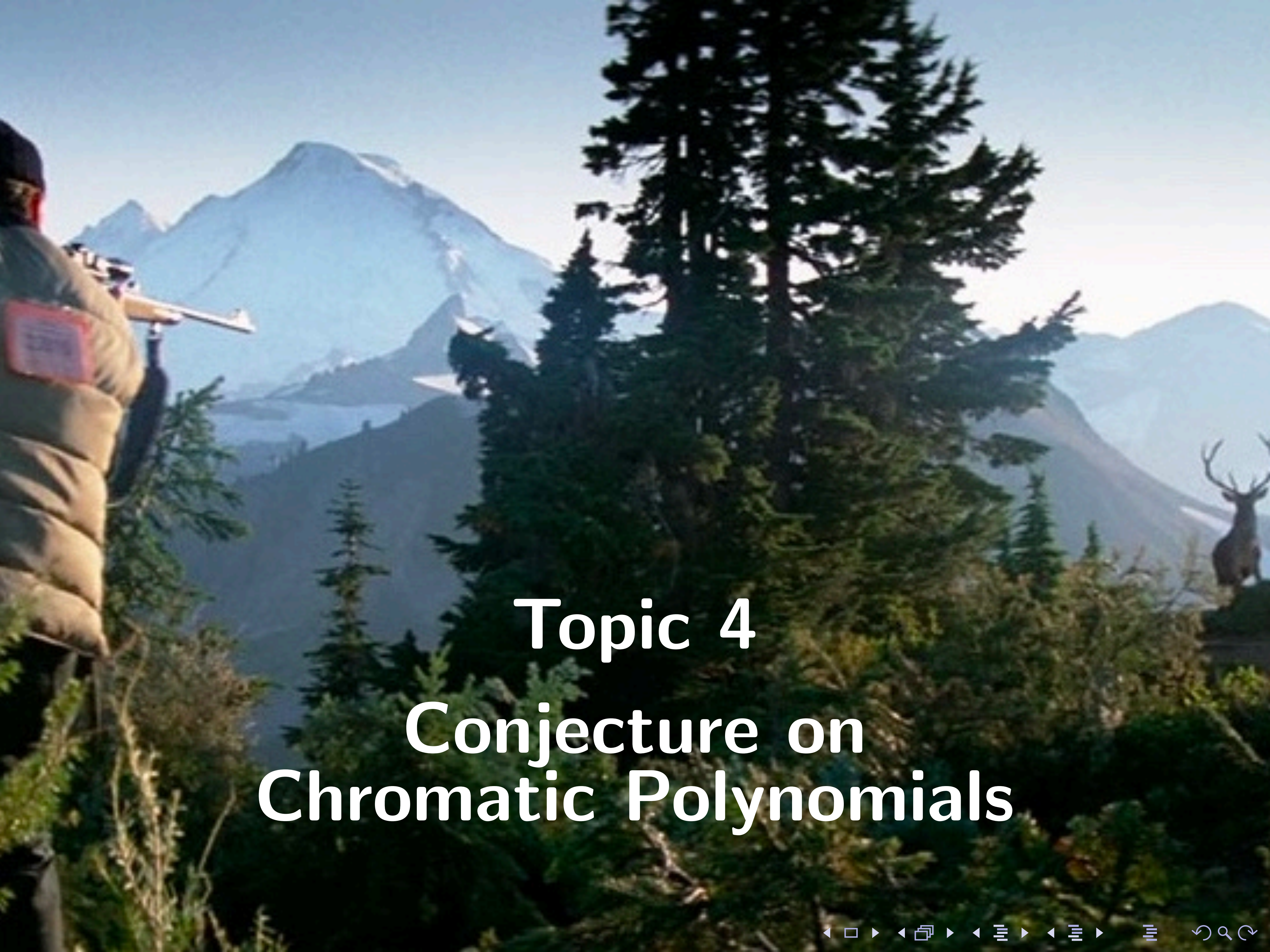
$$H(z,t) = \sum_{n=0}^{\infty} H_n(t) z^n = \frac{(1-t)f(z)}{f(tz) - tf(z)}$$

$$A(z,t) = \sum_{n=0}^{\infty} A_n(t) z^n = \frac{(1-t)f(tz)}{f(tz) - tf(z)}$$

$$G(z,t) = \sum_{n=0}^{\infty} G_n(t) z^n = \frac{(1-t)f(tz)f(z)}{f(tz) - tf(z)}$$

$f(z) = e^z$

- For $a_k = e_k(x_1, x_2, \ldots)$ and $a_k = h_k(x_1, x_2, \ldots)$, these have been studied by e.g. Stanley, Brenti, Stembridge, Shareshian and Wachs.



Topic 4

Conjecture on Chromatic Polynomials

- Conjecture. Let $\chi_G(t)$ be the chromatic polynomial of a graph on $n+1$ vertices. Then

$$t^{n+1} - \chi_G(t) = \sum_{k=1}^{n} h_k \cdot t^{n-k}(t)_k,$$

where $h_k \geq 0$ for all k, and $(t)_k = t(t-1)\cdots(t-k+1)$.

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where $h_k \geq 0$ for all k, and $(t)_k = t(t-1)\cdots(t-k+1)$.
- Theorem (B., Saud, 26). The convex hull of all characteristic polynomials $\chi_{\mathcal{H}}(t)$ of all hyperplane arrangements $\mathcal{H}$ in $\mathbb{F}_q^n$ is a simplex generated by
t^n,
$t^{n-1}(t-1)$,
$t^{n-2}(t-1)(t-q)$,
$t^{n-3}(t-1)(t-q)(t-q^2)$,
$\vdots$
$t(t-1)\cdots(t-q^{n-1})$

Thanks !