The geometry of a counting formula for deformations of the braid arrangement

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Braid-Type Arrangements

Let $\mathbf{S} = (S_{i,j})_{1\le i<j\le n}$ be a collection of finite sets of integers. We define the **S-braid arrangement** $\mathcal{A}_\mathbf{S}$ as the collection of the following hyperplanes:

$$\mathcal{A}_\mathbf{S} := \{H_{i,j,s} \mid 1 \le i < j \le s,\ s \in S_{i,j}\},$$

where $H_{i,j,s} = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_i - x_j = s\}$.

$S_{i,j} = \{-m,\dots,m\}\ \forall i<j$ is the m-**Catalan arrangement**

A counting formula

Bernardi '18: Let $\mathbf{S} = (S_{a,b})_{1\le a<b\le n}$ be a collection of finite sets of integers, and let $\mathcal{A}_\mathbf{S}$ be the **S**-braid arrangement in $\mathbb{R}^n$. Then,

$$\#\text{ regions of } \mathcal{A}_\mathbf{S} = \sum_{(T,B)\in\mathcal{U}_\mathbf{S}(n)} (-1)^{n-|B|},$$

where T is a rooted $(m+1)$-ary plane tree on n labeled nodes and B is a partition of its nodes into "boxes" satisfying certain conditions.

Contribution of a region

Bernardi '18: Bijection between m-Catalan regions and $(m+1)$-ary plane trees on n labeled nodes.

$\mathcal{A}_\mathbf{S}$ is subarrangement of m-Catalan for $m = \max_{s\in S_{a,b}} |s|$. So, for a region R of $\mathcal{A}_\mathbf{S}$, $\mathcal{T}_R$ = the set of trees whose image under this bijection is contained in R, and the **contribution of** R is

$$c(R) = \sum_{\substack{(T,B)\in\mathcal{U}_\mathbf{S}(n)\\ T\in\mathcal{T}_R}} (-1)^{n-|B|}$$

Main result

Theorem:
Let R be a region of the arrangement $\mathcal{A}_\mathbf{S}$. Then $c(R) = 1$.

The Bernardi formula is an easy consequence as then

$$\sum_{(T,B)\in\mathcal{U}_\mathbf{S}(n)} (-1)^{n-|B|} = \sum_{R \text{ region of } \mathcal{A}_\mathbf{S}} c(R) = \#\text{ regions of } \mathcal{A}_\mathbf{S}.$$

Why is this true?

$\mathcal{F}_\mathbf{S}$ = m-Catalan faces that do not lie on any hyperplane in $\mathcal{A}_\mathbf{S}$
$\mathcal{F}_\mathbf{S}(R)$ = faces in $\mathcal{F}_\mathbf{S}$ that lie in R

Theorem: Bijection $\beta : \mathcal{U}_\mathbf{S}(n) \to \mathcal{F}_\mathbf{S}$ with $\dim(\beta(T,B)) = |B|$ that further restricts to a bijection between $\mathcal{U}_\mathbf{S}(R)$ and $\mathcal{F}_\mathbf{S}(R)$.

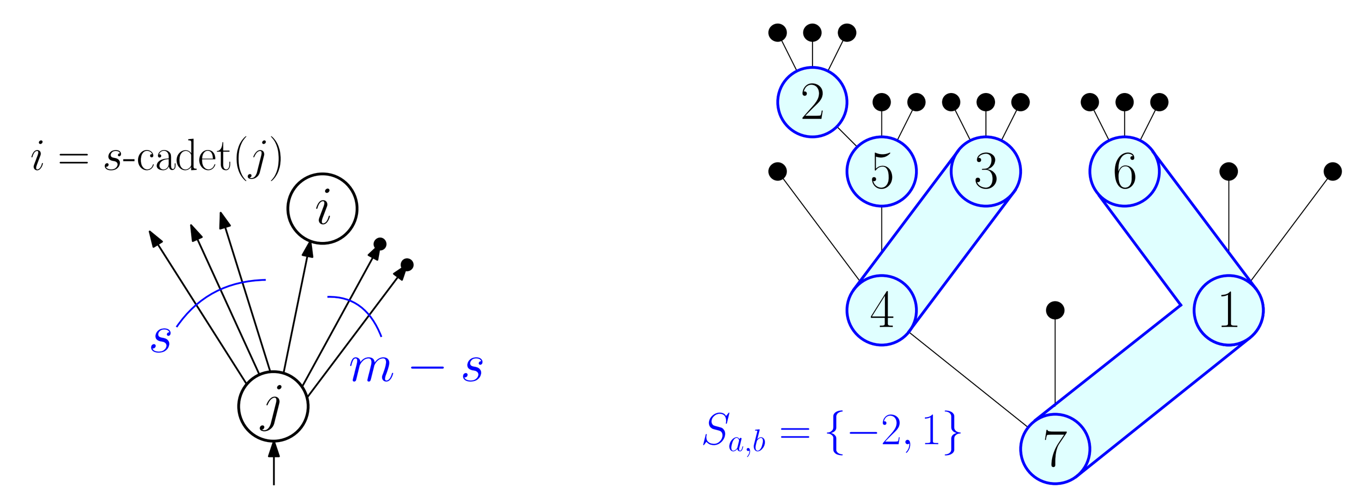
Then, by the bijection,

$$\sum_{(T,B)\in\mathcal{U}_\mathbf{S}(R)} (-1)^{n-|B|} = \sum_{F\in\mathcal{F}_\mathbf{S}(R)} (-1)^{n-\dim(F)}.$$

As the faces in $\mathcal{F}_\mathbf{S}(R)$ partition R, this is the Euler characteristic of R (which is 1 as regions of hyperplane arrangements are contractible).

Boxed Trees

- $\mathcal{T}_n^m$ = set of rooted $(m+1)$-ary plane trees on n labeled nodes
- **cadet**(u) = rightmost non-leaf child of u
- $(v_1,\dots,v_k)$ is an **S-cadet seqence** if $v_{i+1} = s_i$-cadet(v_i), and for $i<j$, for $s = \sum_{p=i}^{j-1} s_p$, the hyperplane $\{x_{v_j} - x_{v_i} = s\} \notin \mathcal{A}_\mathbf{S}$ and if $s = 0$, then $v_i < v_j$.
- A pair (T,B) where $T \in \mathcal{T}_n^m$ and B is a set of **S**-cadet sequences partitioning the nodes of T is a **S-boxed tree**.
- The set of **S**-boxed trees is $\mathcal{U}_\mathbf{S}(n)$



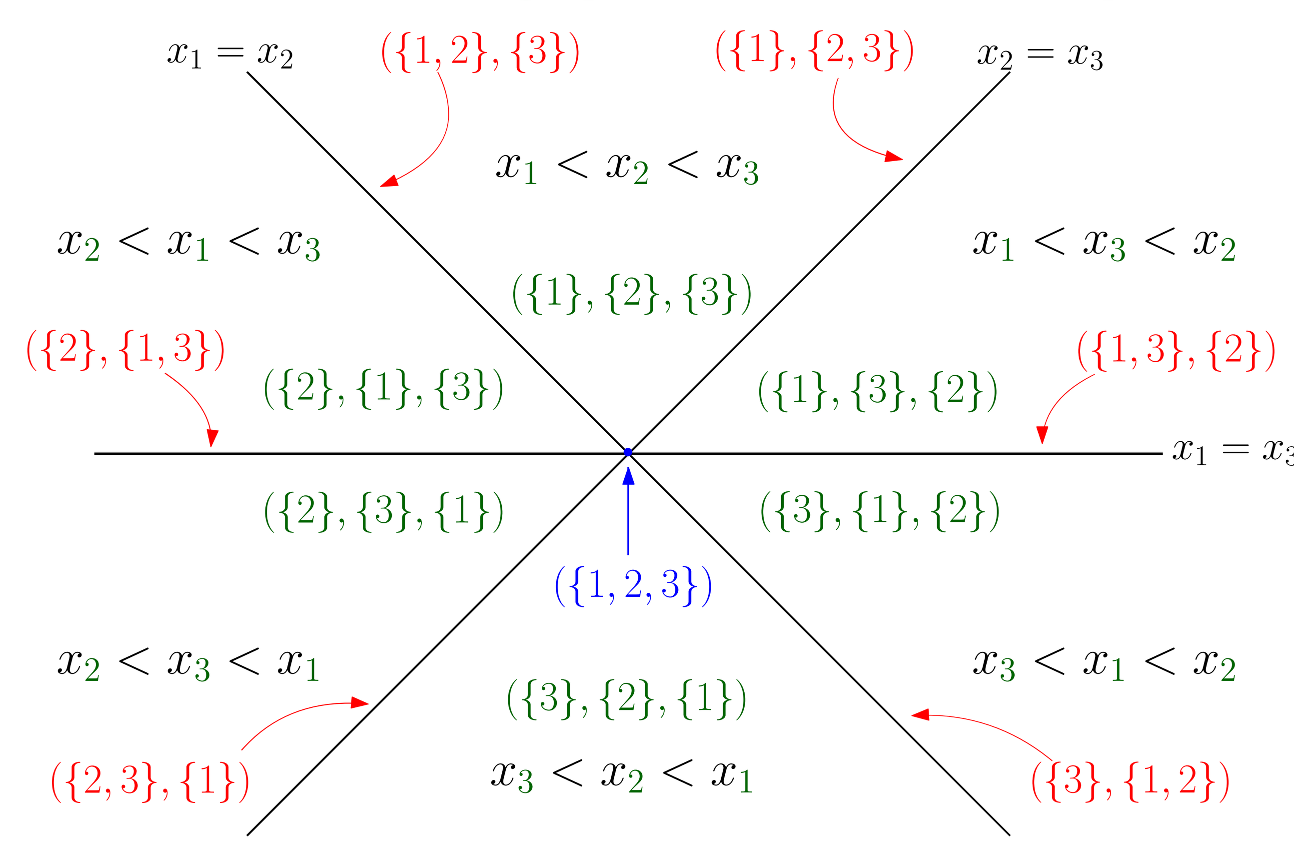
Graphical Arrangements

Let $G = ([n], E)$ be a graph, and $S_{a,b} = \begin{cases}\{0\} & \text{if } \{a,b\}\in E\\ \emptyset & \text{otherwise}\end{cases}$. Then for $S_G = (S_{a,b})$, $\mathcal{A}_{\mathbf{S}_G}$ is a **graphical arrangement**.

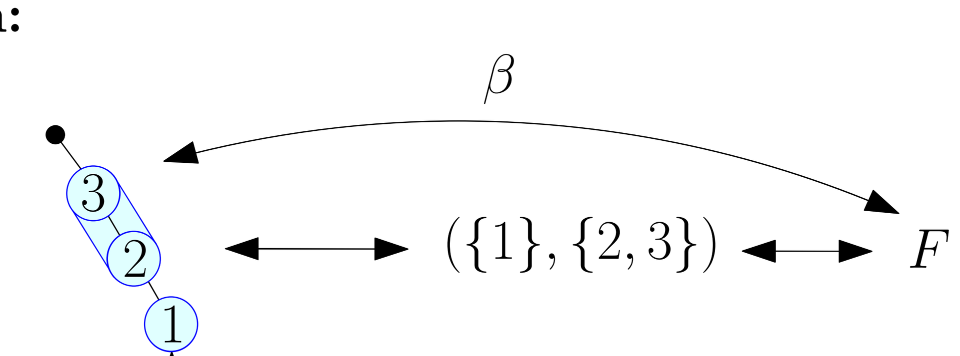
Observe: The trees are unary trees and the $\mathbf{S}_G$-boxed trees can be viewed as ordered partitions of $[n]$ with ordered blocks.

Lemma: $(v_1,\dots,v_k)$ is an $\mathbf{S}_G$-cadet sequence iff $v_i < v_j$ and $\{v_i,v_j\}\notin E\ \forall i<j$.

So: $\mathbf{S}_G$-boxed trees $\leftrightarrow$ ordered partitions of $[n]$ s.t. the induced subgraph on each block is independent



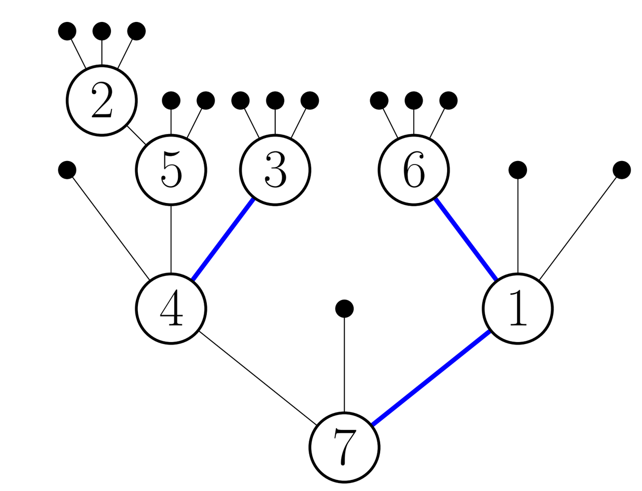
Main Idea:



Braid-Type Arrangements

Bernardi '25: Bijection between m-Catalan faces and marked (m,n)-trees.

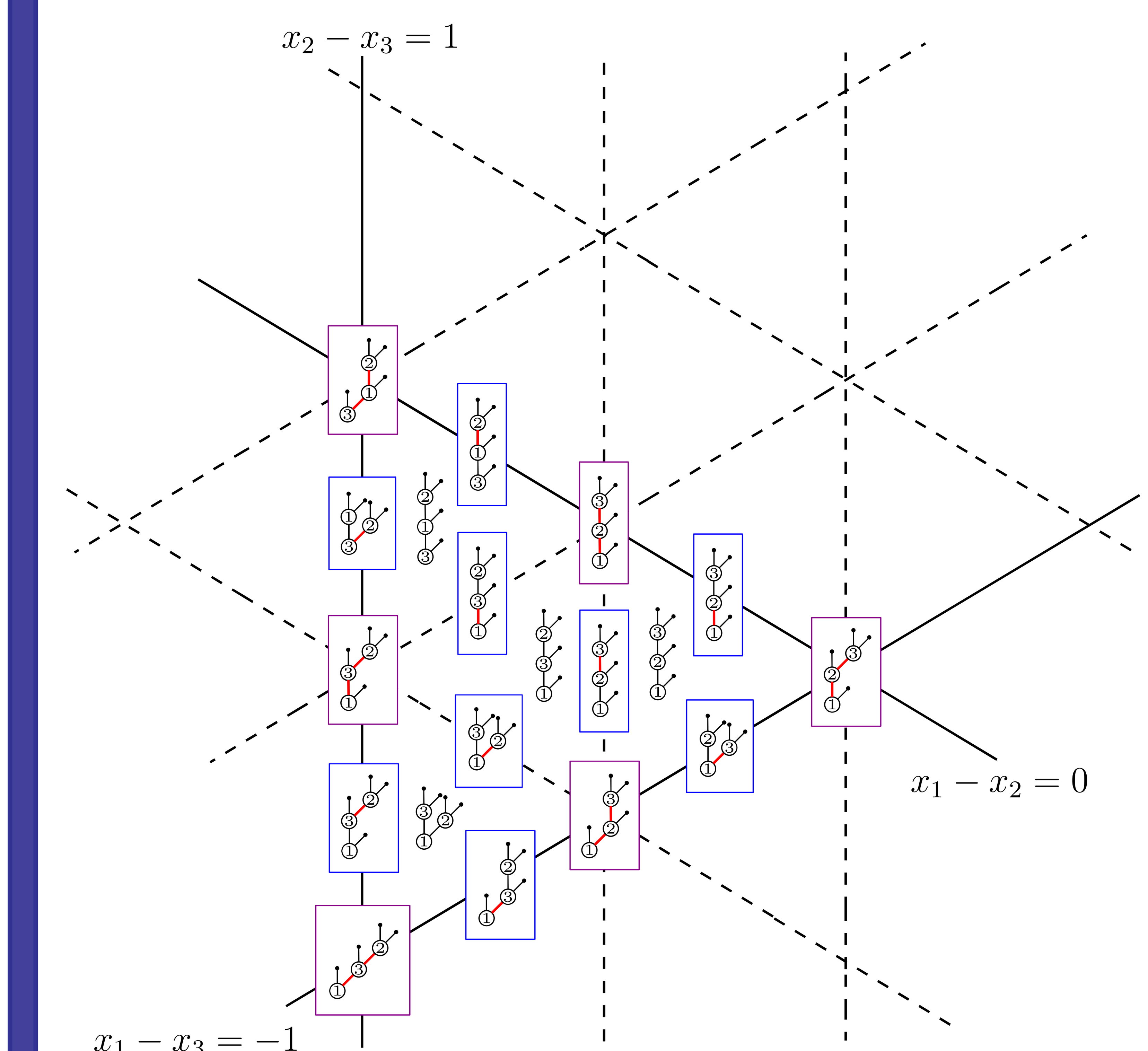
A **marked** (m,n)**-tree** is a pair (T,μ), where $T \in \mathcal{T}_n^m$ and μ is a set of cadet edges such that if $\{j, 0\text{-child}(j)\} \in \mu$ then $j < 0$-child(j).



Important: A marked edge $i = s$-cadet(j) corresponds to the corresponding face lying on the hyperplane $\{x_i - x_j = s\}$.

Main Idea:

- Mark the edges inside a box, compose with Bernardi's bijection
- **S**-cadet condition gives $\beta(T,B) \in \mathcal{F}_\mathbf{S}$
- Avoiding $\mathcal{A}_\mathbf{S}$ hyperplanes guarantees **S**-cadet sequences



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