Rainbow Forests' Color-Coded Obstructions & the 1-System of Curves

Marwa Mosallam & Thomas Zaslavsky
tzaslav@binghamton.edu , mmosallam@binghamton.edu

Research Findings

Obstructions to the existence of total rainbow forests in edge-colored graphs [2].

History and Background

On April 12, 1995, 31 years ago, in the paper named "Systems of Curves on Surfaces" [1], a problem on systems of curves on surfaces was posed by the combinatorists M. Juvan, A. Malnič and B. Mohar where they proved in Theorem 3.3 and 3.4 that for each compact surface with nonempty boundary Σ and each nonnegative integer k there is a constant N with the following property: If Γ is a family of pairwise non-homotopic closed curves on Σ such that any two curves from Γ intersect in at most k points, then Γ contains at most N curves. In our case $k = 1$. So, it was proved that

$$|\Gamma| \leq N(k, g)$$

i.e., the number of curves satisfying the above conditions is finite. Then the same problem was popularized by the geometric topologist Benson Farb where topologists solved the problem for genus 1 & 2 surfaces with a partial progress in genus 3 surfaces.

Obstructions' Ingredients

Theorem 1 (The Fundamental Theorem) *For an edge-colored graph G with n vertices and t colors, G has a total rainbow forest if and only if*

$$r_1(B) \leq r_2(B) \qquad (1)$$

for every absolute set $B \subseteq E(G)$.

Open Question

Question 1 *Find an if-and-only-if criterion for a graph in the setting of Theorem 4 to be a minimal obstruction.*

References

[1] Juvan, M. and Malnič, A. and Mohar, B.: *Systems of curves on surfaces*, Journal of Combinatorial Theory. Series B, 68, , { 7 – 22, 1 },(1996).
[2] Mosallam, M. and Zaslavsky, T.: *Obstructions to Total Rainbow Forests in Edge-Colored Graphs*, arXiv , (2026)
[3] Vondrák, J.: *Lecture notes for "Polyhedral Techniques in Combinatorial Optimization"*, lecture 17 , (2017)
[4] Suzuki, K.: *A necessary and sufficient condition for the existence of a heterochromatic spanning tree in a graph*, Graphs and Combinatorics,22, 261–269, (2006)
[5] Akbari, S. and Alipour, A.: *Multicolored trees in complete graphs*, Journal of Graph Theory,54,3, 221–232, (2007)

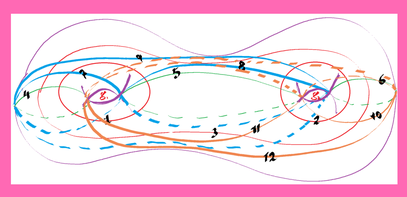
Contribution to literature

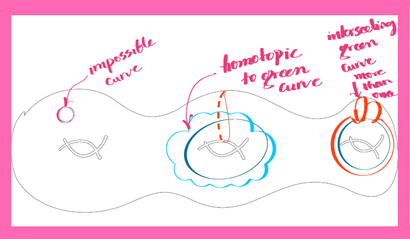
In the Fundamental Theorem, the essential point is that we eliminated the use of A and only need to use $B = E \setminus A$. Suzuki [4], Thrm. 7, has an equivalent theorem (for connected graphs only), stated and proved without matroids , but it is still stated with both sets B and $A = E \setminus B$; as far as we know that is the first appearance of such a theorem. Later, Vondrák [3] stated a version of Suzuki's theorem (without proof or citation) in terms of matroids [3].

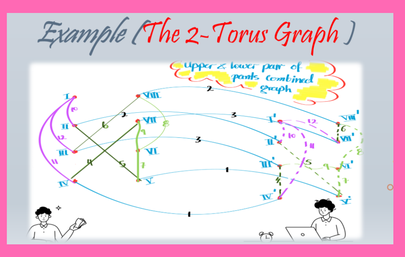
Definitions, Theorems & Examples of Obstructions

Definition 1. Let S_g be a connected, closed, oriented surface of genus g. Let Γ be a set of essential, simple closed curves on the surface S_g, such that:

- No two curves are homotopic.
- Each two curves intersect at at most one point. Call such a set of curves a 1-System of curves on S_g.







Definition 2. In an edge-colored graph, a rainbow forest is a forest whose edges all have different colors. Especially, rainbow forest, whose edges represent every color is a Total Rainbow Forest .

Definition 3. The Rainbow Forest Inequality is defined as follows:

$$\forall A \subseteq E(G): \quad r_1(A) + r_2(E \setminus A) \geq t.$$

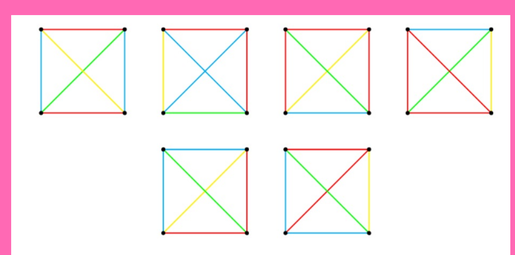
Theorem 2 *Consider an edge-colored graph. Let t be the total number of colors on the edges. For a total rainbow forest to exist, it is necessary and sufficient that the Rainbow Forest Inequality be satisfied.*

Theorem 3 (Edmonds' Matroid Intersection Theorem) *For any two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ we have*

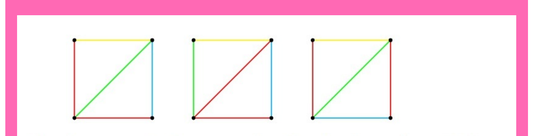
$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{A \subseteq E} [r_1(A) + r_2(E \setminus A)]$$

where r_1 and r_2 are the respective rank functions of M_1 and M_2.

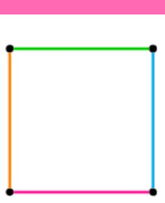
Example: Below is the complete set of minimal obstructions of order 4.



The six minimal obstructions based on K_4.



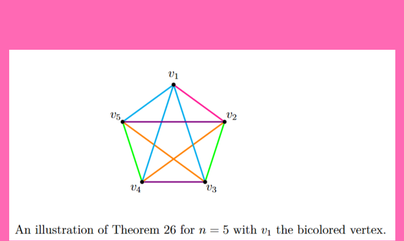
The three minimal obstructions based on the diamond graph $K_4 \setminus e$.



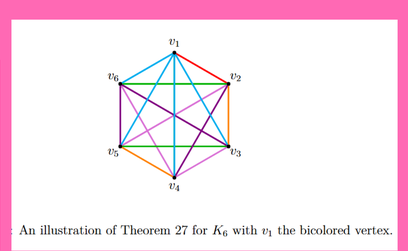
A typical rainbow cycle obstruction

Links to Litrature

Theorem 4 (Bicolored Vertex: Odd Case) *For odd $n \geq 3$, let K_n have n color classes such that one vertex v is incident with all the edges of two classes but no other edges, and the remaining edges are colored in $n-2$ other colors with exactly $\frac{1}{2}(n-1)$ edges in each class. Then this graph is a minimal obstruction.*



An illustration of Theorem 26 for $n = 5$ with v_1 the bicolored vertex.



An illustration of Theorem 27 for K_6 with v_1 the bicolored vertex.

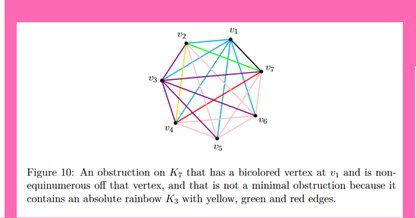


Figure 10: An obstruction on K_7 that has a bicolored vertex at v_1 and is non-equinumerous off that vertex, and that is not a minimal obstruction because it contains an absolute rainbow K_3 with yellow, green and red edges.

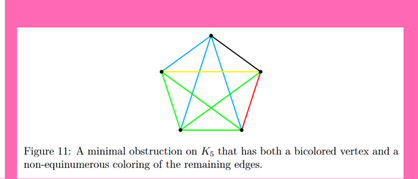
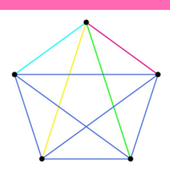


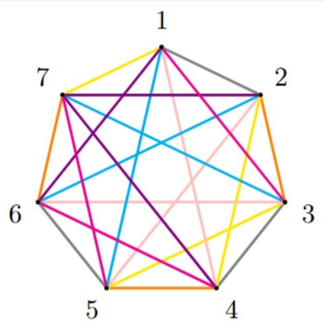
Figure 11: A minimal obstruction on K_5 that has both a bicolored vertex and a non-equinumerous coloring of the remaining edges.

Theorem 5 (Bicolored Vertex: Even Case) *For even $n \geq 3$, let K_n have n color classes such that one vertex v is incident with all the edges of two classes but with no other edges, and the remaining edges are colored in $n-2$ other colors so that each color class has exactly $\frac{1}{2}(n-2)$ or $\frac{1}{2}n$ edges. Then this graph is a minimal obstruction.*

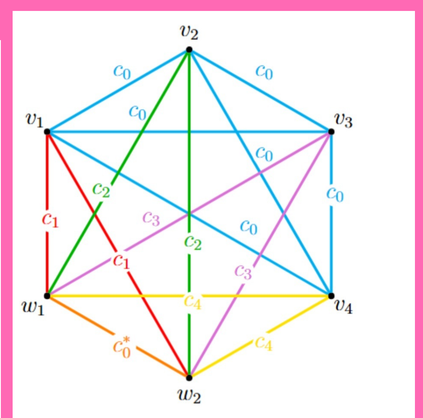
Other Examples of Obstructions



An example of a rainbow-vertex complete graph. The whole graph is the graph G, the blue part of the graph is the graph G_0 and the top vertex is the rainbow vertex v_n where $n = 5$ in this example. In our example here G_0 has no non-adjacencies so no extra edges were added.



An illustration of Pure Equinumerosity for K_7 .



Bipartite Obstruction for $n = 6$.