(Bounded Ratios of)

Principal Minors of Positive Definite Matrices

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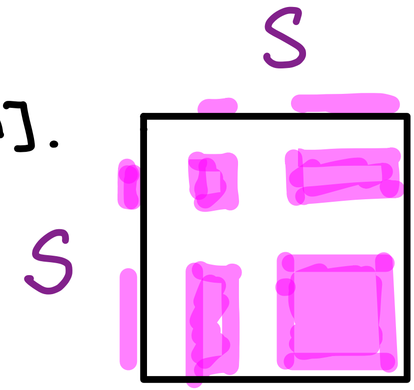
A real symmetric or complex Hermitian $n \times n$ matrix A is called <u>positive definite</u> if

- $x^* A x > 0$ for all nonzero $x \in \mathbb{R}^n$ or $\mathbb{C}^n$
- all eigenvalues > 0
- all principal minors > 0
- $A = B^* B$ for some invertible matrix B

Principal Minors

Let A be an $n \times n$ matrix, $S \subseteq [n]$.



$$A_S := \det(\text{principal submatrix indexed by } S)$$

Hadamard–Fischer–Koteljanskii Inequalities:
1893 1907 1953

For A positive definite, $S, T \subseteq [n]$,

$$A_S \cdot A_T \geq A_{S \cup T} \cdot A_{S \cap T}$$

Principal minors of PD matrices are log-submodular.

This is an example of a bounded ratio:

$$\frac{A_{S\cup T} \cdot A_{S\cap T}}{A_S \cdot A_T} \leq 1$$

Q: Is there a ratio of products of principal minors which is bounded for real symmetric PD matrices but not for complex Hermitian PD matrices?

ref. Boege – Bouthat. Sharp Inequalities for Products of Principal minors of PD matrices

The Cone of Bounded Ratios

For $X \subseteq \mathbb{R}^n_{>0}$, define the set of bounded ratios on X:

$$BR(X) := \{ a \in \mathbb{R}^n : x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \leq c \ \ \forall x \in X \text{ for some } c > 0 \}$$

__Observe__: $BR(X)$ is a cone!

Eg. $e_{S \cup T} + e_{S \cap T} - e_S - e_T \in BR(PM(PD_n))$.

$$BR(PM(PD_n)) \supseteq -(SUBMOD_n)^\vee$$

$$SUBMOD_n = \{ f : 2^{[n]} \to \mathbb{R} \mid f(S) + f(T) \geq f(S \cup T) + f(S \cap T) \}$$

$BR(PM(PD_4))$ was computed by Hall-Johnson (2008)

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Observe: For $x_1, \dots, x_n, c > 0$,

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \leq c \iff \vec{a} \cdot (\log_t x_1, \dots, \log_t x_n) \leq \log_t c$$

$$\log_t c \xrightarrow{t \to \infty} 0$$

Def. For $X \subseteq \mathbb{R}^n_{>0}$

$$\operatorname{trop} X = \lim_{t \to \infty} \{ (\log_t x_1, \dots, \log_t x_n) : x \in X \}$$

Observe: $a \in BR(X) \Rightarrow a \cdot y \leq 0 \ \ \forall y \in \operatorname{trop} X$.

Def. For $X \subseteq \mathbb{R}^n_{>0}$

$$\operatorname{trop} X = \lim_{t \to \infty} \{(\log_t x_1, \ldots, \log_t x_n) : x \in X\}$$

Prop

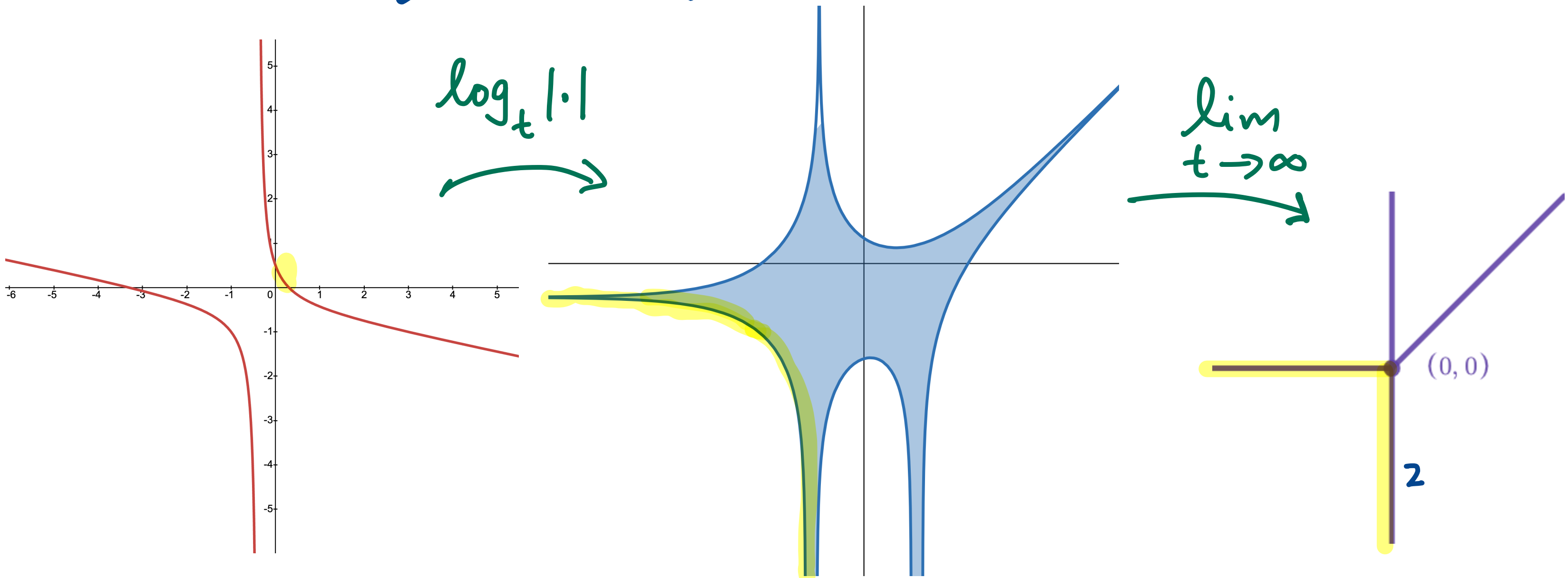
For a semialgebraic $X \subseteq \mathbb{R}^n_{>0}$,

$$BR(X) = -(\operatorname{cone} \operatorname{trop} X)^\vee.$$

In particular, $BR(X)$ is a rational polyhedral cone.

For $X \subseteq \mathbb{C}^n$

$$\operatorname{trop} X = \lim_{t\to\infty} \left\{ (\log_t |x_1|, \ldots, \log_t |x_n|) : x \in X \cap (\mathbb{C}^*)^n \right\}$$



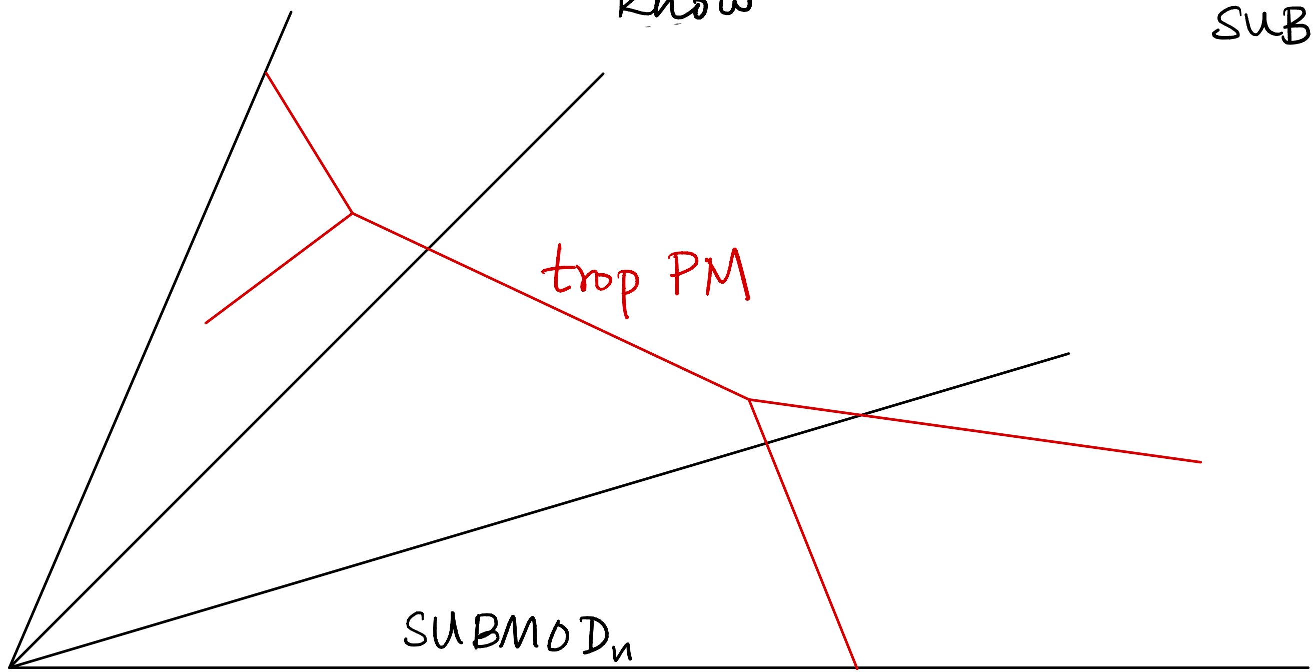
$1 - 3x - x^2 - 2y - 5xy = 0$

<u>Fact</u>: For (semi) algebraic X, $\operatorname{trop} X$ is a rational polyhedral fan.

Question:

$$\mathrm{cone}(\mathrm{trop}(PM(PD_n^{\mathbb{R}}))) \underset{\text{know}}{\subseteq} \mathrm{cone}(\mathrm{trop}(PM(PD_n^{\mathbb{C}})))\ ?$$

know $\supseteq$ $SUBMOD_n$



Theorem (ARVY)

$\mathrm{trop}(PM(PD_n^{K}))$ — $\mathbb{R}$ or $\mathbb{C}$

$=$ tropical complete flag variety over K $\cap$ SUBMOD_n

Brandt–Eur–Zhang "Tropical Flag Varieties" (2020):

For $n \leq 5$, $\mathrm{trop}\, Fl_n^{\mathbb{R}} = \mathrm{trop}\, Fl_n^{\mathbb{C}}$.

<u>Cor:</u> For $n \leq 5$, $BR(PM(PD_n^{\mathbb{R}})) = BR(PM(PD_n^{\mathbb{C}}))$

Question:

$$\mathrm{cone}(\mathrm{trop}(PM(PD_n^{\mathbb{R}}))) = \mathrm{cone}(\mathrm{trop}(PM(PD_n^{\mathbb{C}})))\,?$$

$\subseteq$ know

know $\subseteq$ $SUBMOD_n$

- any counter-example lives in $n \geq 6$, 2^n-dim'l space!
- not enough to check $\mathrm{trop}_{\mathbb{R}} \neq \mathrm{trop}_{\mathbb{C}}$. The cones over them must differ.

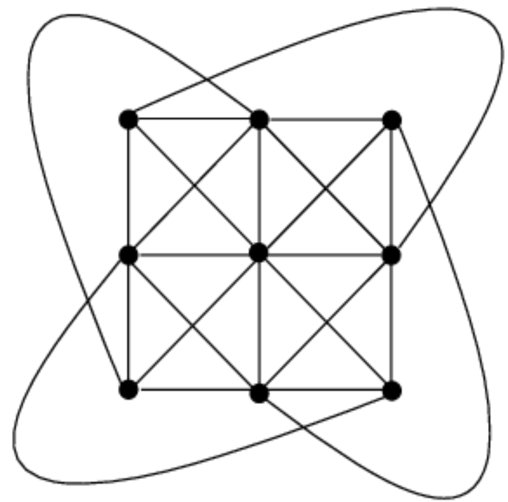
We know:

$$\mathrm{trop}(PM(PD_n^{\mathbb{R}})) \subseteq \mathrm{trop}(PM(PD_n^{\mathbb{C}})) \subseteq \mathrm{SUBMOD}_n$$

<u>IDEA</u>: Look for an extreme ray of SUBMOD_n which is in the tropical variety over $\mathbb{C}$ but not over $\mathbb{R}$.

<u>Fact</u>: Rank functions of connected matroids are extreme rays of SUBMOD.
(but not all extreme rays are this form)

Ternary affine plane matroid $\mathbb{F}_3^2$



representable over $\mathbb{C}$ but not over $\mathbb{R}$

rank 3 on 9 elements

Prop: Let K be an infinite field.
Let M be a matroid on $[n]$.
Then M is representable over K $\iff$ its rank function belongs to $\operatorname{trop}(Fl_{K\{\{t\}\}}(n))$.

Cor: $BR(PM(PD_9^{\mathbb{R}})) \neq BR(PM(PD_9^{\mathbb{C}}))$.

<u>Conclusion</u>: There exists a ratio of principal minors which is bounded for 9×9 real symmetric PD matrices but unbounded for complex Hermitial ones.

But we don't have an explicit description.

Need: find a linear hyperplane separating rank of ternary affine plane from $\operatorname{trop} PM(PD_9^{\mathbb{R}})$