Generalized chip firing and critical groups of arithmetical structures on trees

Kassie Archer, Alexander Diaz-Lopez, Darren Glass, and **Joel Louwsma**

Generalized chip firing

Let G be a finite, connected graph without loops.

If we can find $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^{V(G)}$ and $\mathbf{r} \in (\mathbb{Z}_{>0})^{V(G)}$ for which

$$(\operatorname{diag}(\mathbf{d}) - A(G))\mathbf{r} = \mathbf{0},$$

then the pair $(\mathbf{d}, \mathbf{r})$ gives an *arithmetical structure* on G.

The *critical group* $\mathcal{K}(G;\mathbf{d})$ is the torsion part of the cokernel of the matrix, i.e.

$$\mathbb{Z}^{V(G)} / \operatorname{im}(\operatorname{diag}(\mathbf{d}) - A(G)) \cong \mathbb{Z} \oplus \mathcal{K}(G;\mathbf{d}).$$

Subtracting a column of $(\operatorname{diag}(\mathbf{d}) - A(G))$ corresponds to a firing operation in which the number of chips at each adjacent vertex is increased by 1 and the number of chips at the fired vertex is decreased by the entry of $\mathbf{d}$.

This preserves the *weighted degree* $\mathbf{r} \cdot \delta$ of a chip configuration.

Example

An arithmetical structure on C_4 is given by

$$\mathbf{d} = (3,2,1,4) \qquad \text{and} \qquad \mathbf{r} = (1,2,3,1),$$

with

$$\operatorname{diag}(\mathbf{d}) - A(G) = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix}.$$

The chip configuration $(3,1,-1,-2)$ has degree

$$1 \cdot 3 + 2 \cdot 1 + 3 \cdot (-1) + 1 \cdot (-2) = 0.$$

By firing once at vertex v_1, we obtain the chip configuration

$$(3,1,-1,-2) - (3,-1,0,-1) = (0,2,-1,-1).$$

Connection between chip firing and invariant factors

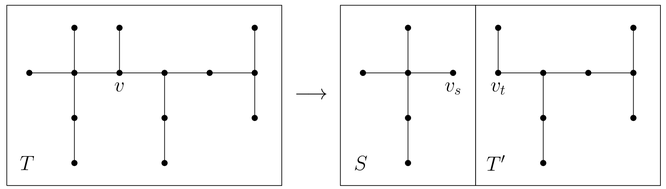
Theorem: Let G be a finite, connected graph without loops. Fix a nonempty subset $Z \subseteq V(G)$ and an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on G. If every chip configuration on G is equivalent (with respect to $\mathbf{d}$) to a chip configuration that is nonzero only at vertices in Z, then $\mathcal{K}(G;\mathbf{d},\mathbf{r})$ has at most $|Z| - 1$ invariant factors.

However, there are examples where not all chip configurations are equivalent to a configuration that is nonzero at only two vertices but the critical group is nevertheless cyclic.

Starlike splittings of trees

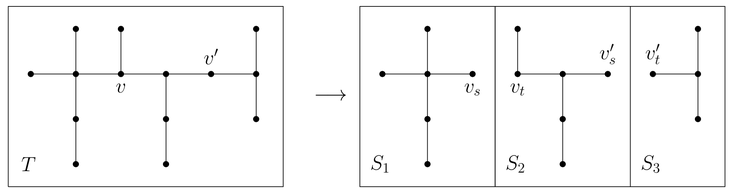
A *starlike tree* is a tree that has exactly one vertex of degree at least 3.

If a tree T has multiple vertices of degree at least 3, a *starlike splitting* of T is a 4-tuple (S, T', v_s, v_t), where S is a starlike tree, T' is a tree, v_s is a leaf of S that is adjacent to its central vertex, v_t is a vertex of T', and T is obtained from S and T' by merging the vertices v_s and v_t.



Starlike decompositions of trees

A *starlike decomposition* of a tree T is a collection $\{S_i\}_{i=1}^k$ of trees with at most one vertex of degree at least 3 that result from a sequence of starlike splittings.



Bounding the number of invariant factors

Theorem: Let T be a finite tree with at least 2 vertices, let $\{S_i\}_{i=1}^k$ be a starlike decomposition of T, and let $\ell_i = \ell(S_i)$.

1. The number of invariant factors of the critical group of any arithmetical structure on T is at most $\sum_{i=1}^k (\ell_i - 2)$.
2. For every integer t satisfying $0 \leq t \leq \sum_{i=1}^k (\ell_i - 2)$, there is an arithmetical structure on T whose critical group has t invariant factors.

Corollary: If T is a finite tree, then the critical group of every arithmetical structure on T is cyclic if and only if

1. T has does not have non-adjacent vertices of degree at least 3, and
2. T has no vertices of degree at least 4.

Splitting irregularity number

A starlike splitting is *regular* if v_t has degree 1 in T' and *irregular* otherwise.

The *splitting irregularity number* of T, denoted $\iota(T)$, is the number of irregular splittings in a starlike decomposition (which is independent of the decomposition).

Theorem: Let T be a finite tree with at least 2 vertices.

1. The number of invariant factors of the critical group of any arithmetical structure on T is at most $\ell(T) - 2 - \iota(T)$.
2. For any integer t satisfying $0 \leq t \leq \ell(T) - 2 - \iota(T)$ there is an arithmetical structure on T whose critical group has t invariant factors.

Constructions

Theorem: Let T be a finite tree. If $\mathcal{G}$ is a finite abelian group with at most $\ell(T) - 2$ invariant factors, then there is an arithmetical structure on a subdivision of T whose critical group is $\mathcal{G}$.