On Thresholds for Spanning Trees

with Caleb Fong & Jinyoung Park

Players: n: large

$G = G_{n,p}$ ("Erdős-Renyi" RG)

T: n-vertex tree

$\Delta = \Delta_T$ (max degree)

Question: when is G likely to contain T?

- when: $p = p_n(\Delta) = p(\Delta)$ (or $p(T)$...)
- likely: n large ("whp")
- contain: copy of

"threshold" [TODAY: happy w order of mag]

$p(\Delta)$ ⟷ when is G likely to contain T?

▶ E.g. $\forall T$ $p(T) \gtrsim \max\{\underbrace{\Delta/n}_{①}, \underbrace{n^{-1}\log n}_{②}\}$

①: TRIVIAL

②: $E(G)$ must cover the vertices

⟷ coupon collector (coupons = vertices)

▶ basic (coupling):

if $\mu := \mathbb{E}\deg(v) \gg \log n$ then

$\deg(v) \approx \mu \ \forall v$ (whp)

$\forall T \quad p(\Delta) \gtrsim n^{-1} \log n$

Conj A (IK '90's, but obv.) $\forall$ fixed Δ

$$p(\Delta) = O(n^{-1} \log n)$$

history: today just:

- Krivelevich ('18) $\forall \varepsilon > 0$

$$p(\Delta) < n^{-1} \left[(k/\varepsilon)\, \Delta \log n + n^{\varepsilon} \right]$$

- Montgomery ('19; $\longleftrightarrow$ Conj A) ?

$$p(\Delta) < K_\Delta\, n^{-1} \log n \qquad (!)$$

Krivelevich: $p(\Delta) < n^{-1}\left[(k/\varepsilon)\,\Delta \log n + n^{\varepsilon}\right]$

Montgomery: $p(\Delta) < K_\Delta\, n^{-1} \log n$

K: ● sharp (up to K) if $\Delta = n^{\varepsilon}$ (WAIT)

M: ● "universal"

● K_Δ: ??

OPEN: K not depending on Δ?

(expect: any $K > 1$)

more recent:

Frankston-K-Narayanan-Park '21
Pham-Park '24
} →

easy pf of: $p(\Delta) < K\Delta n^{-1} \boxed{\log n}$ → "gap"

- vs. Kriuzlevich: $n^{-1}[(k/\varepsilon)\Delta \log n + n^{\varepsilon}]$
- vs. Montgomery: $p(\Delta) < K_\Delta n^{-1} \log n$
  - FKNP/PP not universal

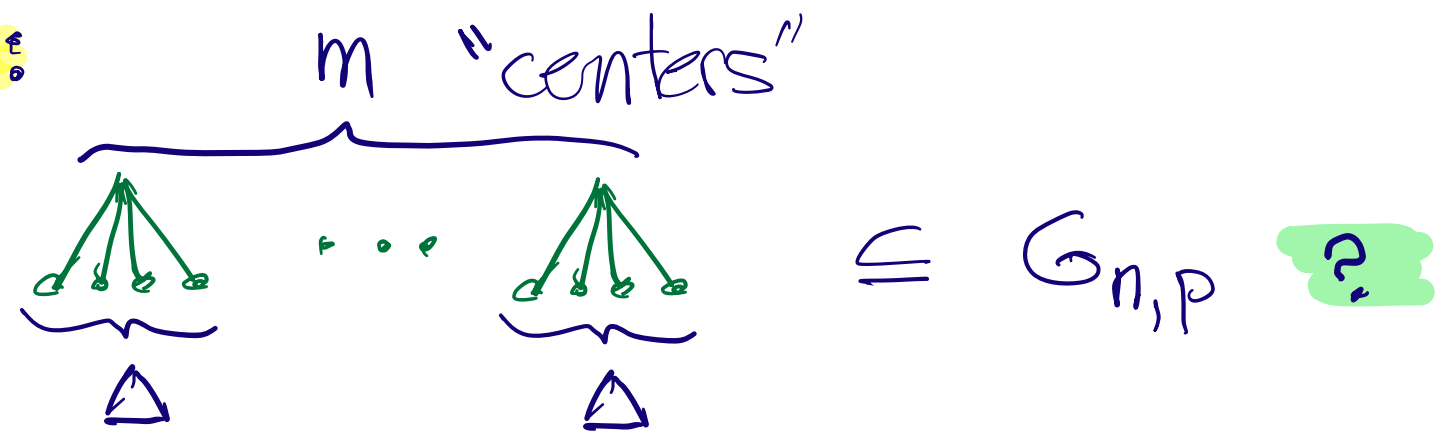
Main Conj (MC) $p(\Delta) < K n^{-1} \max\{\Delta \log \Delta, \log n\}$

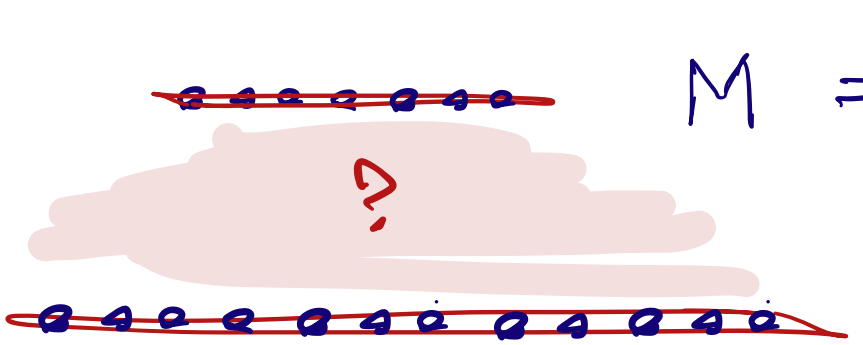
($\overset{\text{e.g.}}{=} K n^{-1} \log n$ if $\Delta < \dfrac{\log n}{\log \log n}$)

cf. Montgomery

MC: $p(\Delta) < K n^{-1} \max\{\Delta \log \Delta, \log n\}$

can't improve:





M = m candidate centers

$V \setminus M$ nbhds must cover $V \setminus M$

Δ

$\rightarrow \binom{n}{m}$ CC problems — must win at least once

MC: $p(\Delta) < K n^{-1} \max\{\Delta \log \Delta, \log n\}$

$\Lambda := \Lambda(T) = \{\text{leaves}\}, \quad \lambda := |\Lambda|$

$\Pi := \{\text{"preleaves"}\}, \quad \pi := |\Pi|$

- Thm 1 true if $\lambda = \Omega(n)$

  e.g. Δ-tree

- Thm 2 true if $\Delta \ll \lambda p$

MC: $p(\Delta) < K n^{-1} \max\{\Delta \log \Delta, \log n\}$

- Thm 1 true if $\lambda = \Omega(n)$
- Thm 2 true if $\Delta \ll \lambda p$

- Thm 2 $\supseteq$ Thm 1 :

$\lambda = \Omega(n) \Longrightarrow p\lambda = \Omega(\Delta \log \Delta)$

... but uses Thm 1 for $\lambda \approx n$ (e.g. Δ-tree)

- Thm 2 $\supseteq$ Montgomery :

not hard ... but uses FKNP/pp (slightly)

proof hints:

an easy case

Thm 1

Thm 2

- An easy case: Δ fixed, $\lambda = \Omega(n)$

start in (any) root & embed (in G):

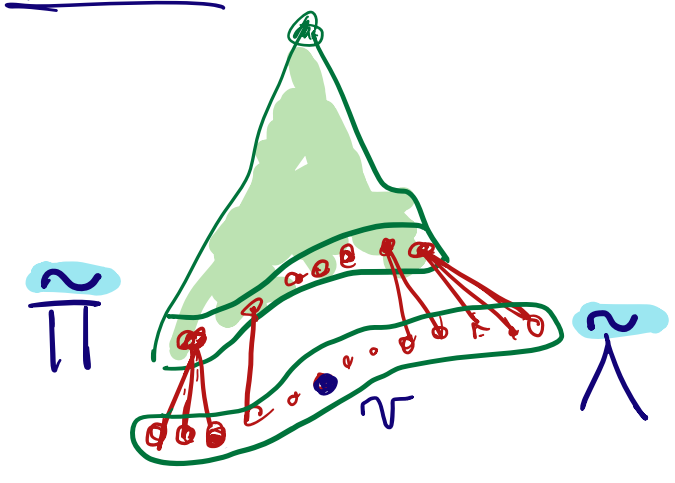
Phase 1 embed $T = \Lambda$ greedily:

$\mathbb{E}\,|\{\text{avail. nbrs}\}| \geq \lambda p \gg \log n$

$\lambda = \Omega(n)$, $p = Kn^{-1}\log n$

← add children

Phase 2



- essentially Hall
- main: $v \in \tilde{\Lambda} \Rightarrow v \sim \tilde{\pi}$

(Δ fixed $\Rightarrow \pi = \Omega(n)$

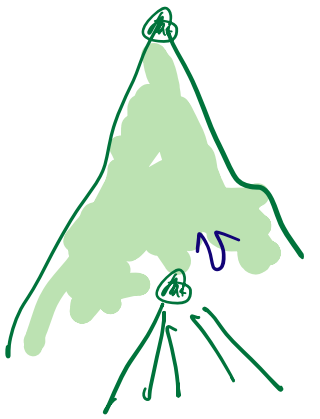
$\Rightarrow \mathbb{E}\, d_{\tilde{\pi}}(v) \gg \log n$)

(not really the picture ...)

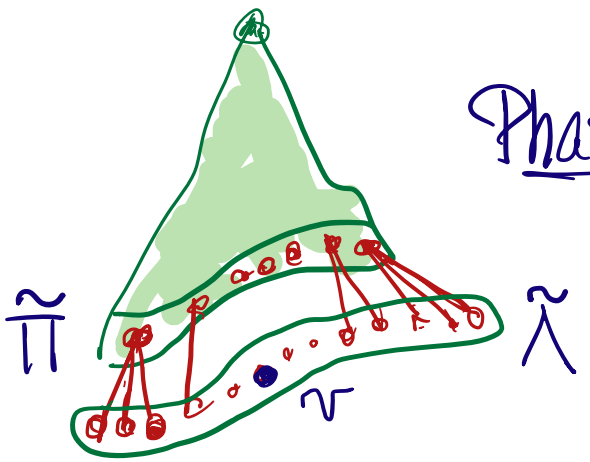
Thm 1 MC time if $\lambda = \Omega(n)$

TRT:

Phase 1: embed $T-\Lambda$ greedily ✓



Phase 2: "Hall" ✗



e.g. $\pi \approx \lambda/\Delta = \Theta(n/\Delta)$ ⇝

$\mathbb{E}\, d_{\tilde{\pi}}(v) = \pi p = \Theta(\log \Delta) \ll \log n$ ⇝

* if $\Delta < n^{o(1)}$

typ: $v \not\sim \tilde{\pi}$ for many $(\Delta^{-c} n)$ v's in $\tilde{\Lambda}$

⟶ ?

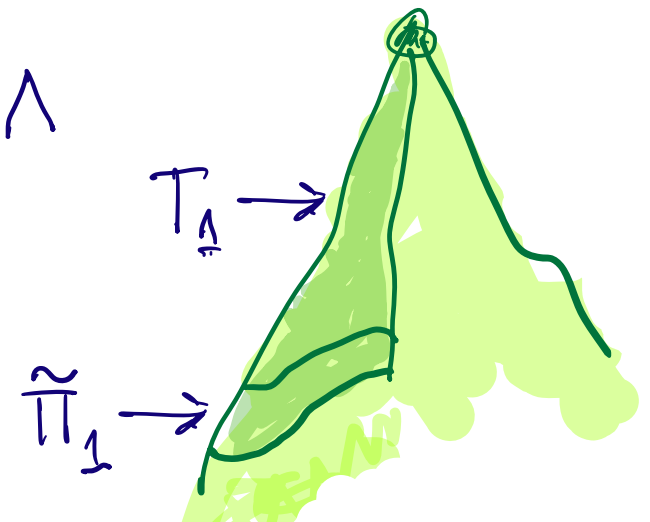
Thm 1 MC time if $\lambda = \Omega(n)$

TRY AGAIN:

Ph 1: greedily embed T_1 = "half" of $T - \Lambda$

$X := \{ v : \text{few nbrs in } \tilde{\pi}_1 \}$

"anemics"



Ph 2: embed rest of $T - \Lambda$ ($\frac{1}{2}$ same as Λ)

- including X

(e.g. Δ-tree 2)

Ph 3: "Hall" (with care)

Thm 2 MC time if $\Delta \ll \lambda p$

(mostly lies :)

Ph 1 : first "third" of T (greedily)

→ Λ tentatively

Ph 2 : 2nd "third" of $T-\Lambda$, with "anemics"

↳ now easy

Ph 3 : rest of $T-\Lambda$

Ph 4 : "Hall"

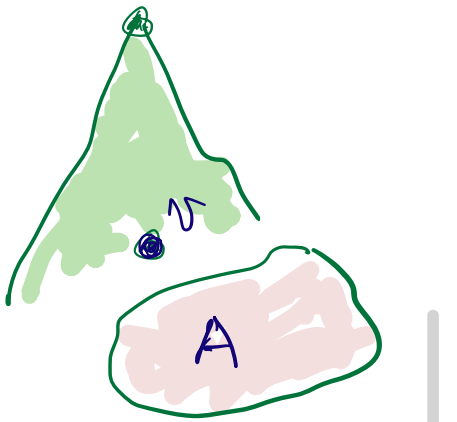
Ph 3: rest of T-A

ISSUE: $|A| \gtrsim \lambda \Longrightarrow$

$\mathbb{E}\, d_A(v) \gtrsim \lambda\rho \gg \Delta$

$d_A(v) \begin{cases} \geq \Delta \\ \neq 0 \end{cases}$

ISSUE: $|A| \gtrsim \lambda$ $\Rightarrow$ $d_A(v) \begin{cases} \geq \Delta \\ \neq 0 \end{cases}$



PLAN (sort of):

- work with D-out ($D \approx np$)
- track "budgets": $b_v = d_A(v)$
- avail. v: reserve for $\tilde{A}$ if b_v becomes small

$\Rightarrow$ embeddeds: always "process" v with b_v min

$\longrightarrow$ RACE: budgets shrink (BAD)
smallest disappears (GOOD)

Thank you