Needles in the Haystack: Finding Nonabelian Partial Difference Sets (PDSs) with Computer Search

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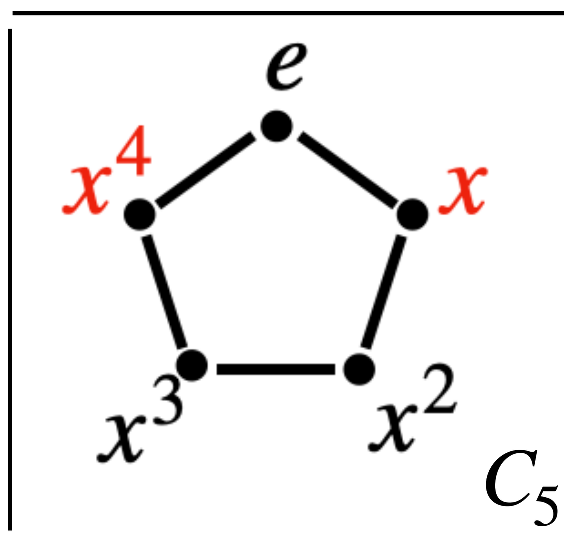
What is a partial difference set (PDS)?

A regular $(n = |G|, k = |D|, \lambda, \mu)$ PDS is a subset $D \subseteq G$ such that $D^2 = ke + \lambda D + \mu(G - D - e)$



Example PDS

$(n, k, \lambda, \mu) = (5,2,0,1)$

$D = x + x^4 \in \mathbb{Z}[C_5]$

$D^2 = (x + x^4)(x + x^4) = 2e + x^2 + x^3$

Negative Latin Square Type

Parameters of form $(n^2, r(n+1), -n + r^2 + 3r, r^2 + r)$

Why Find PDSs?

1. Connections to:
    1. projective 2-weight codes
    2. strongly regular graphs
    3. rank 3 association schemes
2. Testbed for search techniques

What was already known?

- Abelian groups: many constructions known
- Nonabelian groups: comparatively less, though recent work has begun to address gap
- Negative Latin Square Type Nonabelian: very little known
- Challenge: Search space is enormous

Why Nonabelian Groups?

- (Informally) Most PDSs in nonabelian groups [PDSS24]
- Some PDSs (called genuinely nonabelian) are only in nonabelian groups

Method 1: Reducing Search Space with Characters

$\phi : G \to N$, N abelian

$\chi : N \to \mathbb{C}$: character

D is PDS $\Longrightarrow$

$\chi(\phi(D)) \in \text{Spectrum}(D)$

For (64,18,2,6) PDSs:

$N = C_2^3 = \{x^i y^j z^k \mid x^2 = y^2 = z^2 = 1\}$

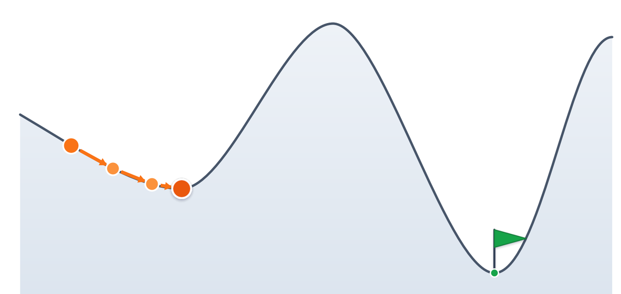
$\text{Spectrum}(D) = \{-6, 2, 18\}$

Number of elements in cosets $(1, x, y, z, xy, xz, yz, xyz)$ is either $(4,2,2,2,2,2,2,2)$, $(4,2,0,2,2,4,2,2)$, etc. Then check all exhaustively.

Results: (64,18,2,6) Search

- Motivation: (64,18,2,6) is negative Latin square type
- Effect of search space reduction $\approx 10^{15} \to 10^{10}$ candidate sets/group
- Main execution time was 3.5 hours on a desktop (code in Java). Preprocessing done in GAP [GAP21].
- Searched exhaustively among groups with C_2^3 normal subgroups (except C_2^6) (212/267 groups of order 64)
- Found precisely <u>48 nonabelian groups</u> with (64,18,2,6) PDSs
- For comparison, <u>only 3 abelian groups</u> with (64,18,2,6) PDSs

Method 2: Hill Climbing Local Search



$\text{Error}(D) = \|D^2 - ke - \lambda D - \mu(G - D - e)\|_2^2$

- Plan: run many instances of hill climb, hope one finds a PDS (ie random restart).
- No additional group-theoretic knowledge used.
- Benefit: Fast
- Drawback: Local search cannot show nonexistence

Results: Local Search

- Local search ran among all possible parameter sets where $n \leq 215$, finding many previously known PDSs, including all found with Method 1
- Found **(147,66,25,33)** and **(144,52,16,20)** PDSs. These
    1. **Proved existence of strongly regular graphs (SRGs) with these parameters**
    2. Are genuinely nonabelian
- (144,52,16,20) is also negative Latin square type
- Brouwer then applied switching on the (147,66,25,33) SRG to get a (148,77,36,44) SRG, which also was previously unknown [Bro24]

	Group	Trials	Time (s)	# PDSs found
(144,52,16,20)	$C_3 \times (C_4^2 \rtimes C_3)$	100K	88.79	3
(147,66,25,33)	$C_7^2 \rtimes C_3$	10K	13.77	78

Runtime Benchmark. 72-core machine. C++ code, shared-memory parallelism with Parlay [BAD20].

Open Questions: PDSs

- Some of the (64,18,2,6) PDSs found were decomposable into three (16,6,2) difference sets (i.e., a linking system). Can we use that structure to generate an infinite family?
- Why are some PDSs easier to find than others (as seen in table)?
- Why does local search work at all? Search space is enormous (10^{40})

Future Directions

- Takeaway: Low-resource computational search is surprisingly effective
- Future work: Low-resource computer search for other interesting combinatorial objects
- If low-resource methods achieved results, how far could high-resource methods (LLMs; SAT solving) go?

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