Dynamics and Combinatorics

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Combinatorics at the Confluence

joint with Joshua Frisch (UCSD)

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Topological dynamics

A **topological dynamical system** (X, T):

- a non-$\varnothing$ **compact** topological space X,
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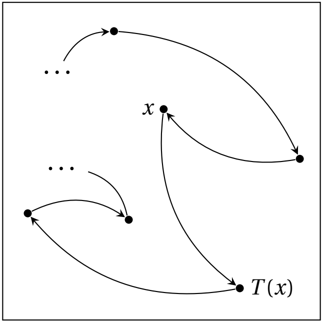
The **orbit** (or "trajectory") of a point $x \in X$: the set $\{T^n(x) : n \in \mathbb{Z}\}$.



Minimal transformations

A transformation $T\colon X \to X$ is **minimal** if every orbit is dense:

For all $x \in X$ and non-$\varnothing$ open $U \subseteq X$, there is some $n \in \mathbb{N}$ such that

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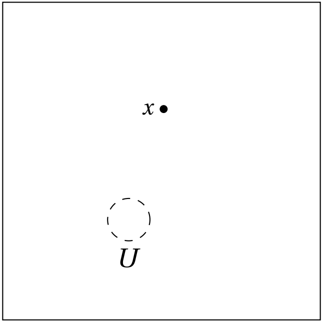

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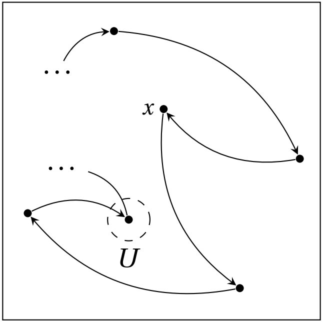

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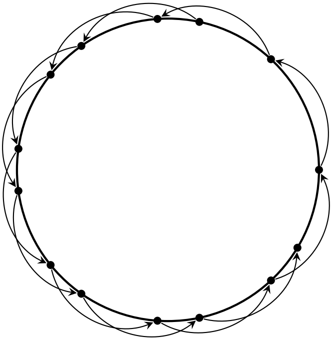

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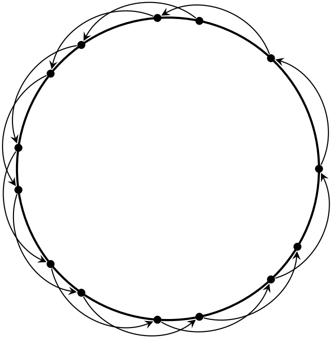

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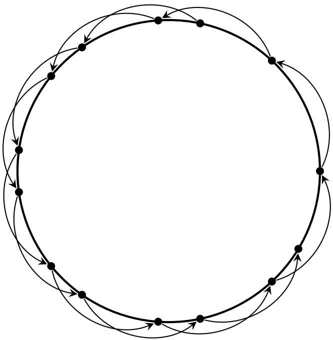

Minimal $\iff$ $\alpha/\pi \notin \mathbb{Q}$ (DIRICHLET 1840s)

Examples

$$X = \text{the torus } \mathbb{R}^2/\mathbb{Z}^2, \quad T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod{\mathbb{Z}^2}$$

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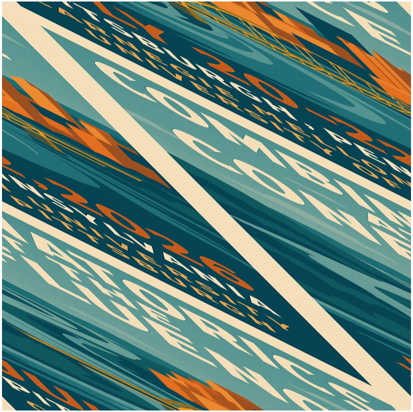

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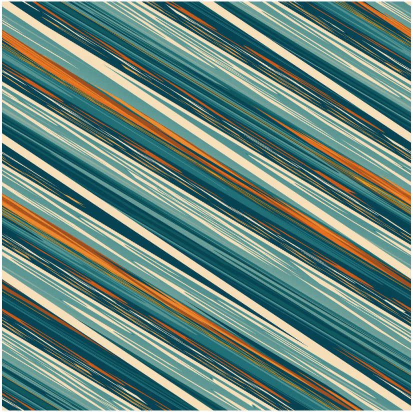

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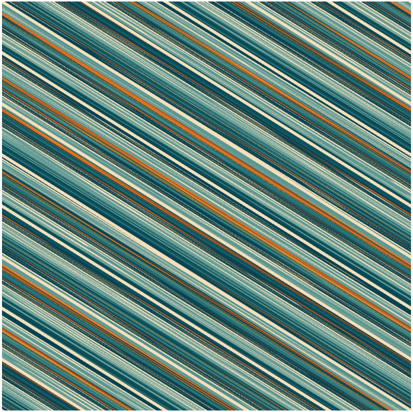

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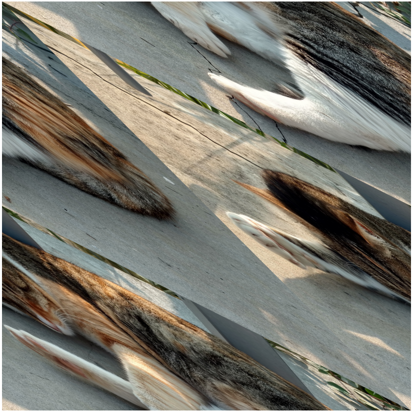

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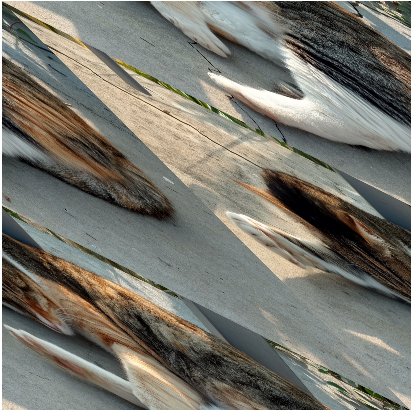

Arnold's cat map. Not minimal (has a fixed point), although almost all orbits are dense.

Minimality and syndeticity

Observation

If $T\colon X \to X$ is minimal, then every non-$\varnothing$ open $U \subseteq X$ is **syndetic**: there is some $n \in \mathbb{N}$ such that each $x \in X$ gets into U within n steps:

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X is **compact** $\Longrightarrow$ get a finite subcover

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- faithful if for all $\mathbf{1} \neq g \in G$, T_g is not the identity map,
- free if for all $\mathbf{1} \neq g \in G$, T_g has no fixed points.

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Fix a countable group G.

Problem ("minimal subdynamics")

Is there a non-degenerate (say, free) G-flow $(X, (T_g)_{g \in G})$ such that T_g **is minimal** for all elements $g \in G$ of infinite order?

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Theorem (AB–Frisch '25+)

The answer is **YES**.

Main result

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Gromov's curse

Every statement about all countable groups is either trivial or false.

Use **combinatorics** instead of group theory!

Symbolic dynamics

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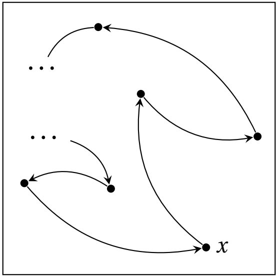

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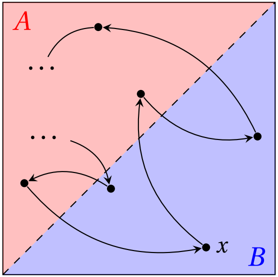

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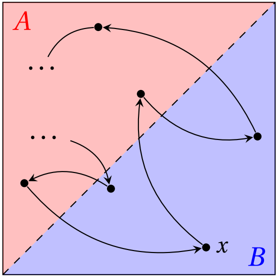

Given $x \in X$, generate an infinite word $w \in \{\mathtt{A}, \mathtt{B}\}^{\mathbb{Z}}$:

$$w \quad = \quad \ldots\,\mathtt{AABABBABA\underline{B}ABAAAABAB}\ldots$$

Record whether each $T^n(x)$ is in A or in B.

Symbolic dynamics

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We say that u occurs in w **syndetically**.

Combinatorics on words

A word $w \in \{\mathtt{A}, \mathtt{B}\}^{\mathbb{Z}}$ such that **every** finite word u that occurs in w does so **syndetically**.

We also want w to be aperiodic (to avoid degenerate solutions).

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Any such word w gives rise to a certain free minimal system (X, T).

General groups

To find a free G-flow $(X, (T_g)_{g \in G})$ such that T_g is minimal, need to solve an analogous combinatorial problem:

Find $w \in \{\mathtt{A}, \mathtt{B}\}^G$ such that **every** finite “pattern” u that occurs in w does so **syndetically** with respect to g.

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Intuition: the probability that a fixed finite word u does not occur in some interval of length n in a random word $w \in \{\mathtt{A}, \mathtt{B}\}^{\mathbb{Z}}$ is exponentially small in n.

The details…

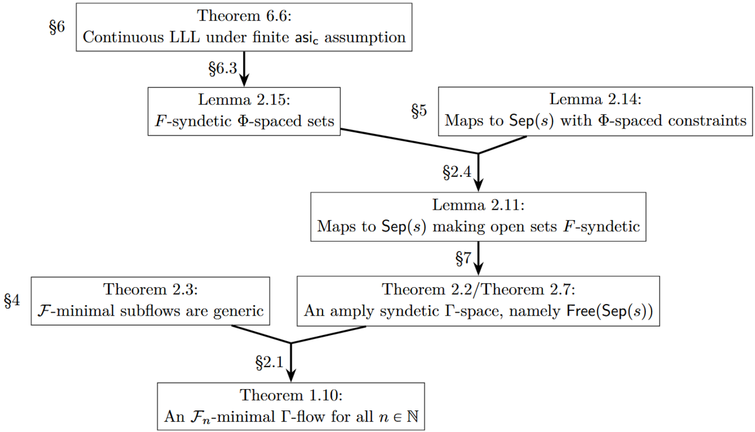

Figure 2. A flowchart for the proof of Theorem 1.10.

Pittsburgh connections: build on

Conley–Jackson–Marks–Seward–Tucker-Drob ’23, AB–Weilacher ’25

Summary

- Many natural questions in dynamics are actually purely combinatorial.
- We can use combinatorial tools to solve them! Especially the probabilistic method.
- Avoid relying on group theory and work in complete generality.
- Surely, many applications of combinatorics in other fields are also waiting to be discovered!

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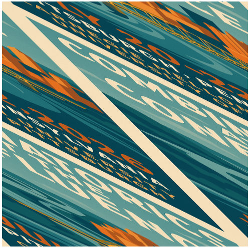

Thank you!